On Robustness of Average Inflation Targeting*

Seppo Honkapohja, Aalto University School of Business
Nigel McClung, Bank of Finland

April 16, 2021

Abstract

This paper considers the performance of average inflation targeting (AIT) policy in a New Keynesian model with adaptive learning agents. Our analysis raises concerns regarding robustness of AIT when agents have imperfect knowledge. In particular, the target steady state can be locally unstable under learning if details about the policy are not publicly available. Near the low steady state with interest rates at the zero lower bound, AIT does not necessarily outperform a standard inflation targeting policy. Policymakers can improve outcomes under AIT by (i) targeting a discounted average of inflation, or (ii) communicating the data window for the target.

Keywords: Adaptive Learning, Inflation Targeting, Zero Interest Rate Lower Bound.

JEL codes: E31, E52, E58

1 Introduction

The past 13 or so years have been a very challenging time for macroeconomic policy-making and in particular for monetary policy. In most years since 2008 central banks have had to keep the policy interest rates at approximately zero level, popularly called the zero lower bound (ZLB) or the liquidity trap.

*We are grateful to Klaus Adam and Markus Haavio for comments. First author acknowledges financial support from the Yrjö Jahnsson Foundation. The views expressed in this paper are not necessarily the views of the Bank of Finland.

The usual framework of inflation targeting became largely ineffective in the ZLB regime which was the initial monetary policy response to the global financial crisis. Once policy rates were effectively down to the ZLB level, central banks had to revert to unconventional monetary policies which took the form of liquidity operations, credit easing, large scale asset purchases and forward guidance about future course of policy. A number of empirical studies have shown that these new policies had qualitatively the right kind of macroeconomic effects, but the estimated magnitudes have been variable.[1]

The ZLB regime after the financial crisis soon inspired discussions among prominent central bankers and academics about possible reform of the monetary policy framework. Alternatives to inflation targeting were explored. Price level targeting (PLT) and the related concept of nominal GDP targeting were perhaps the most widely discussed suggestions for a more appropriate monetary policy framework. Evans (2012) and Carney (2012) were among the first commentators and more recently, for example, Williams (2017), Bernanke (2017) and Bullard (2018) suggested that PLT and more complex "switching policies" should be studied further.

In 2019 the Federal Reserve initiated a review of its monetary policy strategy. The review process culminated in August 2020 in the announcement by Chairman Powell (2020) that the policy framework of the Fed is to be based on Average Inflation Targeting (AIT).[2]

AIT has not been widely studied in the research literature. The paper by Nessen and Vestin (2005) uses a simple average of finite number of lagged inflation rates in a quadratic CB loss function in a standard model with rational expectations (RE). Reifschneider and Williams (2000) suggested a book-keeping device to keep track of deviations between the actual interest rate and a reference rate based on the Taylor rule. Mertens and Williams (2019) use the optimal policy rule under discretion and RE (when the ZLB can be binding) as a reference rate which is combined with AIT. Budianto, Nakato, and Schmidt (2020) propose that the central bank loss function incorporates an exponential moving average of actual inflation rates and they show how the outcome under RE is in welfare terms between the outcomes

[1] See Moessner, Jansen, and de Haan (2017), Dell'Ariccia, Rabanal, and Sandri (2018), Kuttner (2018) and Bhattarai and Neely (2016) for reviews of the empirical literature on unconventional monetary policy.

[2] See e.g. Svensson (2020) for a wide-ranging discussion of alternatives that were considered for the Fed.

from IT and PLT.[3] Hebden, Herbst, Tang, Topa, and Winker (2020) and Andrade, Gali, Le Bihan, and Matheron (2021) are two recent applied papers that focus on alternative make-up strategies for monetary policy.[4]

In this paper we consider the performance of AIT in a standard New Keynesian (NK) model when private agents have imperfect knowledge of the economy and have to engage in learning to forecast its dynamics. It is assumed that when forming expectations private agents statistically estimate the laws of motion for the endogenous variables that they need to forecast.

In this setting private agents make in each period optimal decisions given the current forecasts and the economy reaches a temporary equilibrium for given forecasts and private decisions. As time progresses, new data leads to updating of the forecast functions and a new temporary equilibrium. This approach is called adaptive learning, and it relies on much weaker and arguably more realistic assumptions than rational or boundedly rational decision rules under RE.

The induced learning behavior influences the actual dynamics of endogenous variables. In benign circumstances the economy reaches a long-run RE equilibrium, but this depends on the structure of the economy and in particular the policy rule used by the central bank. The convergence to long run equilibrium does not necessarily take place and the form of monetary policy can play a crucial role in guaranteeing long run convergence.[5]

Our analysis raises warning signals as regards robustness of economic performance with AIT policy in conditions of imperfect knowledge and learning. As a starting point it is assumed AIT is practiced under opacity of its details, which is arguably the current framework of the Fed.[6] In this setup the outcome is precarious in that local convergence of learning to target steady state depends on stickiness of prices. With full price flexibility there is typically local instability while with price stickiness the steady state is locally stable but only if the speed of learning is very low.

---

[3] They also allow for bounded rationality in the form of ad hoc deviations from optimal private decision rules under RE.

[4] There are also other studies by the staff of the Federal Reserve System that were done as part of the Fed strategy review.

[5] Evans and Honkapohja (2003) and Evans and Honkapohja (2006) showed how the form of interest rate rule for implementing optimal policy can be crucial to ensure convergence to REE. Bullard and Mitra (2002) studied convergence conditions when a Taylor rule is applied.

[6] See e.g. the interview of John Williams in FT Live, November 13, 2020.

Modifications to the AIT setup are considered next. It turns out that exponential discounting of old data in a finite window can somewhat improve the properties of AIT under opacity. Giving more information to private agents about the structure of AIT is a superior way to improve the chance of a stable outcome with AIT policy. With the latter assumption AIT can sometimes initiate an escape from the ZLB/liquidity trap regime. However, AIT may also induce deflationary spirals near the low steady state in cases where even a standard inflation targeting Taylor rule would initiate escape from the ZLB. Thus, AIT does not ensure a better outcome at the ZLB than a standard inflation targeting policy.

In the next section we introduce the AIT formulation and the basic results using two simple models. The main model and results are developed in Section 3. Sections 4 and 5 consider the alterations to the AIT framework. Proofs of the results and various modelling details are in the several appendices.

2 Introductory Examples

The basic idea behind average inflation targeting (AIT) is that, when comparing actual inflation against its long run target, the measure of actual inflation is an average of past inflation rates. This average inflation rate is the key indicator for policy decisions, so that policy is tightened vs. loosened if the average of past inflation rates is above vs. below respectively its target value. There are of course different ways to measure average value and there is also the issue of length of the data window in its computation. In addition, major issues about communication of the AIT framework to the private economy are relevant.

A natural starting point is that there is opacity about the details of the AIT rule. We now develop two very simple examples to illustrate that introducing AIT under opacity can create stability concerns in the resulting economic dynamics.

2.1 Fisherian model

The first example is the simplest form of what is called the classical monetary model[7]. The model includes flexible prices and perfect competition among

[7] See e.g. chapter 2 of Gali (2008).

identical households. The latter optimize an intertemporal utility function over streams of consumption and employment for an infinite horizon. There are also identical perfectly competitive firms who produce a single good using labor as the only productive input. Prices are flexible, there is market clearing and real variables are determined independently of monetary policy. Determination of nominal variables, inflation and nominal interest rates, depends the conduct of monetary policy, where the latter is assumed to be based on an interest rate rule.

The Fisher equation which follows from intertemporal consumption optimality is the key relation to link the real interest rate to inflation and nominal interest rate. We start with the form

$$R_t = \beta^{-1}\pi_t^e \tag{1}$$

which comes from the consumption Euler equation (assuming, for exposition's sake, constant income and no shocks to household preferences). Here R_t is the gross nominal interest rate, $\pi_t^e = (p_{t+1}/p_t)^e$ is expected gross inflation rate π_{t+1} in period $t+1$ and formed in period t (note the unusual notation!). β is the subjective discount rate, so that β^{-1} is the gross real rate of interest. Equation (1) is taken to describe the interest rate desired by private agents for given inflation expectations.[8] Its linearization around the target steady state is

$$\hat{R}_t = \beta^{-1}\hat{\pi}_t^e,$$

where hat notation, e.g. $\hat{R}_t$, denotes a linearized variable as deviation from target steady state.

The second equation of the model describes AIT monetary policy. The nominal interest rate is set in response to an average of deviations from inflation target π^* in the past $L-1$ periods.[9] Its linearization around the steady state with π^* is

$$\hat{R}_t = \frac{\psi}{\pi^*}\sum_{k=0}^{L-1}\hat{\pi}_{t-k}, \tag{2}$$

where $\pi^* < \beta\psi$ is assumed. Derivation of (2) is discussed in more detail in the beginning of Section 3.1. Combining the linearization of (1) and (2)

[8] This formulation corresponds to what is called Euler equation learning in which agents have short decision horizon, see e.g. Honkapohja, Mitra, and Evans (2013) and Evans and Honkapohja (2013).

[9] The inflation term could be divided by L, but this would not change the result as L can be incorporated into ψ.

yields the temporary equilibrium relation as an implicit equation that determines $\hat{\pi}_t$ for given expectations $\hat{\pi}_t^e = E_t^* \hat{\pi}_{t+1}$ and lags $\hat{\pi}_{t-i}$, $i = 1, ..., L-1$, where E_t^* denotes (possibly) non-rational expectations. After combining the linearization of (1) and (2) and rearranging, the system is

$$\hat{\pi}_t = \frac{\pi^*}{\beta\psi}\hat{\pi}_t^e - \sum_{i=1}^{L-1} \hat{\pi}_{t-i}. \tag{3}$$

As there is opacity about monetary policy, it is assumed that private agents do not know anything about the interest rate rule (2) and therefore the agents forecast inflation using a simple weighted average of past inflation, called steady state learning with constant gain which formally is

$$\hat{\pi}_t^e = \hat{\pi}_{t-1}^e + \omega(\hat{\pi}_{t-1} - \hat{\pi}_{t-1}^e), \tag{4}$$

where $\omega > 0$ is a small constant. Stability is taken to mean convergence for all sufficiently small $\omega > 0$. (See Section 3.2 for more discussion of learning.)

We are interested in local convergence of the system (3)-(4) for different lengths of the data window $L-1$. The result is:

Remark: *Assume that $\pi^* < \beta\psi$. The steady state π^* is locally stable under the system (3)-(4) if $L \leq 3$ but is explosive if $L = 4$ and for many higher values of L.*

Figure 1 illustrates the instability with $L = 5$ and standard numerical values for the parameters $\pi^* = 1.005$, $\beta = 0.99$, $\psi = 1.5$ and $\omega = 0.001$. The initial conditions are $\hat{\pi}_0^e = 0.02$, $\hat{\pi}_0 = 0.09$, $\hat{\pi}_{-1} = 0.1$, $\hat{\pi}_{-2} = 0.001$, $\hat{\pi}_{-3} = 0.03$ and $\hat{\pi}_{-4} = 0.05$. Divergence is very slow as the gain parameter ω is very small. Figure A.2 in the Appendix gives a very long simulation to verify divergence.

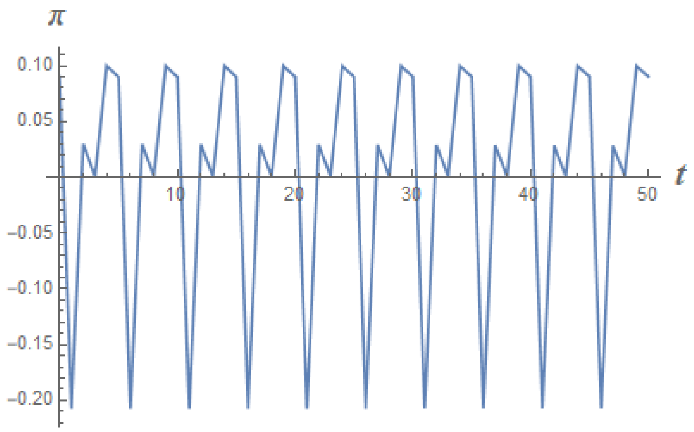


Figure 1: Unstable dynamics with AIT in Fisher model.

This remark can be proved by computing the characteristic polynomial to see when its roots are inside the unit circle, see Appendix C.2. We will give detailed proof using a more general model with flexible prices.

To get an intuition for this result we do the following operations to the model. Start by lagging (3) one period and solve for $\hat{\pi}_{t-1}$. Then combine the result, equation (4) and (3) which yields the equation

$$\hat{\pi}_t = \omega(\frac{\pi^*}{\beta\psi} - 1)\hat{\pi}_{t-1} - \omega \sum_{i=2}^{L-1} \hat{\pi}_{t-i} + (1-\omega)\hat{\pi}_{t-L}. \quad (5)$$

When the gain parameter is very small ($\omega \to 0$), equation (5) becomes $\hat{\pi}_t \approx \hat{\pi}_{t-L}$. Under a simple inflation targeting rule ($L = 1$) equilibrium inflation is a monotonic sequence that converges slowly to steady state if $\pi^* < \beta\psi$ and $\omega > 0$ is small. With AIT ($L > 1$), the sequence of inflation is no longer monotone. For example, the AIT rule may aim to overshoot the target ($\hat{\pi}_t > 0$) after an initial undershooting ($\hat{\pi}_0 < 0$), which can trigger a subsequent tightening of policy and undershooting of inflation as incoming high inflation data replaces the low initial inflation data in the measure of average inflation.

The resulting oscillatory dynamics are unstable if (1) long run inflation expectations $\hat{\pi}_t^e$ drift up (down) in periods of high (low) inflation because agents do not understand the temporary nature of these oscillations (policy is set under opacity); (2) L is large, such that the periods of over- and undershooting are long. Thus, for $L > 3$, (5) is an unstable process for any $\omega > 0$.

We note in Section 3.1 that as $L \to \infty$, the AIT rule becomes a Wicksellian price level targeting (PLT) rule which is a stable process for inflation, provided $\pi^* < \beta\psi$. Thus, unstable dynamics emerge under AIT with opacity for sufficiently high L, but not under IT or PLT.

2.2 Simple Model with Sticky Prices

To illustrate the role of price stickiness, we consider a log-linearized New Keynesian (NK) model with Euler equation learning.[10] The standard IS equation for household decision making is

$$\hat{y}_t = \hat{y}_t^e - \sigma(\hat{R}_t - \hat{\pi}_t^e), \tag{6}$$

where $\hat{y}$ is the output gap, $\hat{\pi}$ is inflation, $\hat{R}$ is the nominal interest rate, $\hat{\pi}_t^e = E_t^* \hat{\pi}_{t+1}$ and $\hat{y}_t^e = E_t^* \hat{y}_{t+1}$. $\hat{\pi}$ and $\hat{y}$ variables are log-deviations from the target steady state. The central bank is assumed to set interest rates using the log-linearized AIT rule

$$\hat{R}_t = \psi \sum_{k=0}^{L-1} \hat{\pi}_{t-k}. \tag{7}$$

Inflation expectations $\hat{\pi}_t^e$ evolve according to (4). The Phillips curve equation is of standard form

$$\hat{\pi}_t = \beta \hat{\pi}_t^e + \kappa \hat{y}_t \tag{8}$$

and we assume output expectations are based on (8) and given as[11]

$$\hat{y}_t^e = \frac{1-\beta}{\kappa} \hat{\pi}_t^e. \tag{9}$$

If we substitute (9), (8), and (7) into (6) we have

$$\hat{\pi}_t = \frac{\kappa^{-1} + \sigma}{\kappa^{-1} + \sigma\psi} \hat{\pi}_t^e - \frac{\sigma\psi}{\kappa^{-1} + \sigma\psi} \sum_{k=1}^{L-1} \hat{\pi}_{t-k}. \tag{10}$$

Note that in the NK model κ is decreasing in price rigidity and in the limit $\kappa = 0$ implies constant inflation rate while prices become totally flexible as

[10] For references on the concept of Euler equation learning, see footnote 6.

[11] Alternatively, we could assume that $\hat{y}_t^e$ is based on a steady state learning scheme akin to (4), but this would not affect the qualititate results in this section.

$\kappa \to \infty$. In the limit $\kappa \to \infty$ (10) has nearly the same form as (3) so $\hat{\pi}_t$ will oscillate and diverge for large L as was illustrated in the preceding example in Section 2.1.

By lagging (10) one period and solving for $\hat{\pi}^e_{t-1}$, then substituting this expression for $\hat{\pi}^e_{t-1}$ and (4) into (10) we obtain for any κ

$$\hat{\pi}_t = \frac{1-\kappa\sigma\omega(\psi-1)}{\kappa\sigma\psi+1}\hat{\pi}_{t-1} - \frac{\omega\kappa\sigma\psi}{1+\kappa\sigma\psi}\sum_{k=2}^{L-1}\hat{\pi}_{t-k} + \frac{\kappa\sigma\psi(1-\omega)}{\kappa\sigma\psi+1}\hat{\pi}_{t-L}, \quad (11)$$

$$\hat{\pi}^e_t = \omega\hat{\pi}_{t-1} + (1-\omega)\hat{\pi}^e_{t-1}. \quad (12)$$

When prices are very sticky (i.e. κ is small), inflation $\hat{\pi}_t$ is not very responsive to higher lags of inflation. For very small ω we then have $\hat{\pi}_t \approx A\hat{\pi}_{t-1}$ with A slightly smaller than 1 which suggests slowly convergent dynamics. Proposition 1 below shows that the learning dynamics under AIT and opacity are locally convergent in a standard New Keynesian model with price stickiness.

The contrasting results of the two examples suggest that AIT yields stability when there is price stickiness (provided that speed of convergence under learning, ω, is low), but is problematic if the economy has flexible prices.

3 New Keynesian Model

A standard New Keynesian model is employed as the analytical framework which uses the assumption that price stickiness arises from price adjustment costs. See e.g. Benhabib, Evans, and Honkapohja (2014) and Honkapohja and Mitra (2020) for more details.

A continuum of household-firms produce a differentiated consumption good under monopolistic competition and price adjustment costs in the spirit of Rotemberg (1982). The utility and production functions are assumed to be identical and agents have homogenous point expectations, so that there is a representative agent. Government uses monetary policy, buys a fixed amount of output and finances spending by taxes and issues of public debt. For simplicity, consumers are assumed to be Ricardian and monetary policy is conducted in terms of an interest rate rule in the cashless limit. Formal details of the model are in Appendix A.

The analysis relies on two behavioral rules of private agents: the Phillips curve and the consumption function. Starting with the former, the Phillips curve takes the form

$$Q_t = \tilde{K}(y_t, y^e_{t+1}, y^e_{t+2} ...) \equiv \frac{\nu}{\alpha\gamma} y_t^{(1+\varepsilon)/\alpha} - \frac{\nu-1}{\gamma} \frac{y_t}{(y_t - (\bar{g}+\tilde{g}_t))^\sigma} + \tag{13}$$
$$\frac{\nu}{\gamma} \sum_{j=1}^{\infty} \alpha^{-1} \beta^j \left(y^e_{t+j}\right)^{(1+\varepsilon)/\alpha} - \frac{\nu-1}{\gamma} \sum_{j=1}^{\infty} \beta^j \frac{y^e_{t+j}}{(y^e_{t+j} - (\bar{g} + \rho^j \tilde{g}_t))^\sigma},$$

where

$$Q_t = (\pi_t - 1)\, \pi_t, \tag{14}$$

while y_t denotes output and $\bar{g}$, $\tilde{g}_t$ are the mean and random parts of government spending and ν, α, γ and β are parameters for substitution elasticity, labor input exponent in production, price adjustment costs and subjective discount rate, respectively. Prices are fully flexible if $\gamma \to 0$, and otherwise prices are sticky if $\gamma > 0$, see Appendix A.6. Superscript e indicates expectations while subscripts indicate the periods $t+j$, $j = 0, 1, 2, ...$ See Appendix A.2 for details.

Expectations in (13) are formed at time t and based on parameter estimates of the forecast function that uses information about endogenous variables at the end of period $t-1$. Actual variables and the observable exogenous random shock at time t are assumed to be known when agents make current decisions. Equation (13) is treated as one of temporary equilibrium equations that determine π_t, given expectations $\{y^e_{t+j}\}_{j=1}^{\infty}$.[12]

To derive the consumption function it is assumed for simplicity that consumers are Ricardian in the sense that they amalgamate their own intertemporal budget constraint and that of the government (where the latter is evaluated at price expectations of the consumer). It can be shown that the consumption function takes the form

$$c_t \sum_{j=1}^{\infty} \beta^{j/\sigma} (D^e_{t,t+j})^{(1-\sigma)/\sigma} = \sum_{j=1}^{\infty} (D^e_{t,t+j})^{-1} (y^e_{t+j} - (\bar{g} + \rho^j \tilde{g}_t)), \tag{15}$$

where c_t, $\pi_t = P_t/P_{t-1}$ and R_t denote private consumption, (gross) inflation rate and (gross) interest rate for loan from period t to $t+1$. The discount

[12] Note that in the representative agents case expected future inflation rate does not directly affect current inflation. There is an indirect effect via current output in the Phillip's curve (13). Using (39) in the first-order conditions to eliminate relative prices and the representative agent assumption, each firm's output equals average output in every period. Since firms can be assumed to have learned this to be the case, we obtain (13).

factor is

$$D^e_{t,t+j} = \frac{R_t}{\pi^e_{t+1}} \prod_{i=2}^{j} \frac{R^e_{t+i-1}}{\pi^e_{t+i}}. \tag{16}$$

See Appendix A.3 for details. In practice central banks do not make their policy instrument rules known. This is reflected in (16) as private agents form expectations about future interest rates.

It can be noted that if there are no adjustment costs in price setting (i.e. $\gamma = 0$ in (38) in the appendix), then the Phillips curve is replaced by a static first order condition for consumption and labor supply. We continue to analyze the case of price flexibility below.

3.1 Average Inflation Targeting (AIT)

It is initially assumed that there is opacity about the details of monetary policy, so that agents do not know the interest rate rule and they need to forecast future interest rates. The central bank uses an interest rate rule that depends on average inflation targeting

$$\begin{aligned} R_t &= 1 + \max[\bar{R} - 1 + \psi_p[(P_t - \bar{P}_{t,L})/\bar{P}_{t,L}] + \psi_y[(y_t - y^*)/y^*], 0], && (17) \\ \bar{P}_{t,L} &= (\pi^*)^L P_{t-L} \text{ and} && (18) \\ \pi_t &= P_t/P_{t-1}. && (19) \end{aligned}$$

Here $\bar{P}_{t,L}$ denotes the target price level. It is formulated with a target level for inflation π^* and $\bar{P}_{t,L}$ is computed by compounding the actual price level L periods ago using target inflation rate π^*. Notice that (17) becomes a simple inflation targeting rule when $L = 1$. As $L \to \infty$, (17) becomes a Wicksellian price level targeting (PLT) rule with inflation target path given by $\bar{P}_{t,\infty} = (\pi^*)^t P_0$ for all t.

The rule (18) implies that

$$\frac{P_t}{\bar{P}_{t,L}} = \frac{P_t}{P_{t-1}} \cdots \frac{P_{t-(L-1)}}{(\pi^*)^L P_{t-L}} = (\pi^*)^{-L} \prod_{i=0}^{L-1} \pi_{t-i},$$

so the basic AIT rule with the ZLB constraint can be written as

$$\begin{aligned} R_t &= R(y_t, \pi_t, ..., \pi_{t+1-L}) && (20) \\ &\equiv 1 + \max[\bar{R} - 1 + \psi_p \left[(\pi^*)^{-L} \prod_{i=0}^{L-1} \pi_{t-i} - 1 \right] + \psi_y[\frac{y_t}{y^*} - 1], 0]. \end{aligned}$$

Rule (20) is the starting point of our analysis of average inflation targeting, but other variants will also be considered.

3.2 Temporary Equilibrium and Learning

Following the literature on adaptive learning, it is assumed that each agent has a model for perceived dynamics of state variables, also called the perceived law of motion (PLM). In any period the PLM parameters are estimated by recursive least squares using available data and the estimated model is used for forecasting. The PLM parameters are re-estimated when new data becomes available in the next period. In linearized models, a common formulation is to postulate that the PLM is a linear regression model where endogenous variables depend on intercepts, observed exogenous variables and (possibly) lags of endogenous variables.[13] The estimation is based on least squares or related methods.[14]

Under opacity about the monetary policy framework agents must forecast the interest rate as well as output and inflation rate without any knowledge of the variables and their lags in the policy rule. In this situation agents' learning is about how to forecast future inflation, output and interest rate and, as a starting point, agents are assumed to exclude lagged endogenous variables from their PLM. Note that the only lags in the model are lagged inflation rates in the policy rule and private agents have no knowledge of the form (20). With this assumption the equilibrium involves imperfect knowledge and is thus a *restricted perceptions equilibrium*.[15] Details of the formulation are discussed in Appendix A.5.

As discussed in Appendix A.5, stability of a steady state can be validly assessed using the simplifying assumption that the random part of government spending $\tilde{g}_t$ is identically zero. Agents estimate the long run mean values of state variables, called "steady state learning" which is formalized as

$$s^e_{t+j} = s^e_t \text{ for all } j \geq 1, \text{ and } s^e_t = s^e_{t-1} + \omega_t(s_{t-1} - s^e_{t-1}), \tag{21}$$

[13] The assumption of a linear PLM is often used as an approximation also in nonlinear models as the true nonlinear functional form of the model would involve expectations of complicated nonlinear functions.

[14] For discussions of adaptive learning, see e.g. Evans and Honkapohja (2001), Sargent (2008) and Evans and Honkapohja (2009a).

[15] See e.g. Evans and Honkapohja (2001) and Branch (2006). The term self-confirming equilibrium is also used in in the literature, see e.g. Sargent (1999).

where $s = y$, π, R. It should also be noted that in this notation expectations s_t^e refer to future periods (and not the current one) formed in period t. When forming s_t^e the newest available data point is s_{t-1}, i.e. expectations are formed in the beginning of the current period. 'Constant gain' learning is assumed, so that the gain parameter is $\omega_t = \omega$, for $0 < \omega \leq 1$ and assumed to be small. Note that the agents in the simple models of section 2 updated their inflation expectations using a steady state learning scheme with constant gain (see (4) and (12)).

3.3 AIT Monetary Policy under Opacity and Learning

The temporary equilibrium equations of the model with steady-state learning consists of
(i) the infinite horizon Phillips curve (14) with equation (22) below,
(ii) the aggregate demand function coupled with market clearing, equation (23) below, and
(iii) the interest rate rule (20) including the definition of inflation.
Agents form expectations π_{t+j}^e, y_{t+j}^e and R_{t+j-1}^e for $j = 1, 2, ...$ We initially consider the nonstochastic case where agents do steady-state learning[16] given by equation (21) and the model equations become
(i) Phillips curve

$$\begin{aligned}
\pi_t(\pi_t - 1) &= \tilde{K}(y_t, y_t^e) \equiv \frac{\nu}{\alpha\gamma} y_t^{(1+\varepsilon)/\alpha} - \frac{\nu - 1}{\gamma} \frac{y_t}{(y_t - \bar{g})^\sigma} + \\
&\quad \frac{\nu}{\gamma} \sum_{j=1}^{\infty} \alpha^{-1} \beta^j \left(y_t^e\right)^{(1+\varepsilon)/\alpha} - \frac{\nu - 1}{\gamma} \sum_{j=1}^{\infty} \beta^j \frac{y_t^e}{(y_t^e - \bar{g})^\sigma} \text{ or} \\
\pi_t &= \Pi(y_t, y_t^e) \equiv Q_{-1}[\tilde{K}(y_t, y_t^e)], \qquad (22)
\end{aligned}$$

(ii) aggregate demand function

$$\begin{aligned}
y_t &= Y(y_t^e, \pi_t^e, R_t, R_t^e) \qquad (23) \\
&\equiv \bar{g} + \left[\left(\frac{\beta R_t}{\pi_t^e} \right)^{-1/\sigma} \left(1 - \beta^{1/\sigma} \left(\frac{R_t^e}{\pi_t^e} \right)^{(1-\sigma)/\sigma} \right) \right] (y_t^e - \bar{g}) \left(\frac{R_t^e}{R_t^e - \pi_t^e} \right)
\end{aligned}$$

(iii) interest rate rule (20).

[16] Note the timing convention for R_t above.

The system is

$$
\begin{aligned}
y_t - Y(y_t^e, \pi_t^e, R_t, R_t^e) &= 0 \\
\pi_t - \Pi(y_t, y_t^e) &= 0 \\
R_t - R(y_t, \pi_t, ..., \pi_{t+1-L}) &= 0
\end{aligned}
$$

and to analyze E-stability of steady states the system is written in abstract form

$$
F(X_t, X_t^e, X_{t-1}, ..., X_{t-(L-1)}) = 0 \tag{24}
$$

where F consists of the aggregate demand function, the Phillips curve and the interest rate rule. The vector of current state variables is $X_t = (y_t, \pi_t, R_t)^T$ while $(X_{t-1}, ..., X_{t+1-L})^T$ contains the lagged endogenous variables. The rule of steady state learning for the components of X_t can be written in vector form as

$$
X_t^e = (1-\omega)X_{t-1}^e + \omega X_{t-1}. \tag{25}
$$

Local stability properties of steady states under the rule (25) is now analyzed. Linearizing around the target steady state we obtain the system

$$
\begin{aligned}
\hat{X}_t &= (1-\omega)M\hat{X}_{t-1}^e + (\omega M + N_1)\hat{X}_{t-1} + \sum_{i=2}^{L-1} N_i \hat{X}_{t-i}, \text{ where} \quad (26) \\
M &= (-DF_X)^{-1}(DF_{X^e}) \text{ and } N_i = (-DF_X)^{-1}(DF_{X_{-i}}). \quad (27)
\end{aligned}
$$

where the matrices $M, N_1, ..., N_{L-1}$ are given in the Appendix, $\hat{X}_t = (\hat{y}_t, \hat{\pi}_t, \hat{R}_t,)^T$, and the hat denotes a linearized variable. For brevity, unchanged notation for the deviations from the steady state is used. Recall also that X_t^e refers to all expected future values of X_t and not the current one.

The focus is on "small gain" results, i.e. whether stability obtains for all ω sufficiently close to zero.

Definition. The steady state is said to be **expectationally stable** or **(locally) stable under learning** if it is a locally stable fixed point of the system (26) and (25) for all $0 \leq \omega < \bar{\omega}$ for some $\bar{\omega} > 0$.

Conditions for this can be directly obtained by analyzing (25)-(26) in a standard way as a system of linear difference equations. Alternatively, so-called expectational stability (E-stability) techniques based on an associated differential equation in virtual time can be applied, e.g. see Evans and Honkapohja (2001).

The key result is that there is local stability of constant gain learning if there is price stickiness because of adjustment costs:[17]

Proposition 1 *Assume that there is price stickiness ($\gamma > 0$) and $\psi_p > \beta^{-1}\pi^*$, $\nu > 1$ and $\sigma \geq 1$. For small ω, the target steady state is locally stable under constant gain learning under the rule (17) for all L.*

However, the preceding stability result is overturned when there is full price flexibility. Appendix A.6 sketches how the model changes when there is price flexibility (adjustment costs are 0). For that model we have:

Proposition 2 *Assume that there is full price flexibility ($\gamma = 0$) and $\psi_p > \beta^{-1}\pi^*$. For small ω, the target steady state is locally stable under constant gain learning under the rule (17) for $L \leq 3$ but is unstable for many higher values of L.*

The proposition can be proved using the Schur-Cohn formulae which are, however, cumbersome to use for high values of L. We remark that an alternative method of proof based on complex analysis is developed in the Appendix. This method is convenient for high values of L.

Propositions 1 and 2 raise questions about applicability of the results. As stability is overturned in the limit $\gamma \to 0$ to price flexibility, it is imperative to study whether the AIT rule ensures a stable equilibrium for empirically plausible values of the gain parameters when there are positive adjustment costs $\gamma > 0$. An interest rate rule is said to be "robustly stable" if it leads to stable outcomes across a wide range of empirically plausible values of the gain parameter, ω.[18] As will be seen in the first row of Table 1 in Section 4.3, the basic AIT rule does not lead to robust stability as problems can arise even for very small positive values of the gain parameter ω.

An obvious response to this problem is to use discounting of older data in the AIT rule and we study the extension in Section 4.

[17] Proofs are given in the Appendices.

[18] The idea of using the range of values for gain parameter as a criterion for robustness was first suggested in Evans and Honkapohja (2009b).

4 Using Exponential Weighting in AIT

In this section we consider whether use of weighted measures of average inflation that discount past inflation relative to current inflation in computing AIT can improve stability properties of AIT policy.

4.1 Exponentially Declining Weights

Here we introduce exponentially declining weights over the finite past horizon when computing average inflation for the interest rate rule. Thus the rule (20) is adjusted to

$$R_t \equiv 1 + \max[\bar{R} - 1 + \psi_p \left[\frac{1-\mu}{1-\mu^L} \sum_{i=0}^{L-1} \mu^i (\frac{\pi_{t-i}}{\pi^*} - 1) \right] + \psi_y [\frac{y_t}{y^*} - 1], 0], \quad (28)$$

where $0 < \mu < 1$. The length of the past horizon is $L-1$ as before. Note that the weights in the rule sum to 1. The framework is otherwise unchanged: the aggregate demand function (23), the Phillips curve (22) and steady state learning.

The economy is stable under learning with exponentially declining weights, i.e. interest rule (28) under opacity:

Proposition 3 *(i) Assume that there is full price flexibility ($\gamma = 0$) and $\psi_p > \beta^{-1}\pi^*$ and $0 < \mu < 1$. For all small ω, the target steady state is locally stable under constant gain learning under the rule (28) for all L.*
(ii) Assume that there is price stickiness ($\gamma > 0$) and $\psi_p > \beta^{-1}\pi^*$, $\nu > 1$, $\sigma \geq 1$, and $0 < \mu < 1$. For small ω, the target steady state is locally stable under constant gain learning under the rule (28) for all L.

Robustness of stability in Proposition 3 (ii) is examined using a calibrated model in Section 4.3.

4.2 Exponential Moving Average Rule

A different way to discount old data is to assume that an exponential moving average specification is used in the interest rate rule. Consider an interest rate rule that responds to an exponential moving average of inflation:[19]

$$
\begin{aligned}
R_t &= 1+\max[\bar{R}-1+\psi_p\left(\frac{\pi_t^{w_c}(\pi_t^{cb})^{1-w_c}}{\pi^*}-1\right),0] \qquad (29)\\
\pi_t^{cb} &= \pi_{t-1}^{w_c}(\pi_{t-1}^{cb})^{1-w_c} \qquad (30)
\end{aligned}
$$

where $0 < w_c < 1$. The framework is otherwise unchanged: the aggregate demand function (23), the Phillips curve (22) and steady state learning.

The dynamic model is now given by

$$F(X_t, X_t^e, \pi_t^{cb}) = 0 \qquad (31)$$

where F consists of the Phillips curve, the aggregate demand function and interest rate rule (29). The vector of current state variables is $X_t = (y_t, \pi_t, R_t)^T$. The law of motion for X_t^e is the same as before, and the law of motion for π_t^{cb} is given by (30). Linearizing around the target steady state we obtain the system

$$
\begin{aligned}
\hat{X}_t &= (-DF_X)^{-1}(DF_{X^e}\hat{X}_t^e + DF_{cb}\hat{\pi}_t^{cb}) \qquad (32)\\
&\equiv M\hat{X}_t^e + N\hat{\pi}_t^{cb} \qquad (33)
\end{aligned}
$$

where M and N are given in the appendix, and $\hat{X}$ again collects linearized y, π, R. In a model with sticky prices and an exponential moving average rule, the Taylor Principle is now sufficient for stability under constant gain learning:

Proposition 4 *(i) Assume that there is price stickiness ($\gamma > 0$) and $\psi_p > \beta^{-1}\pi^*$, $0 < w_c < 1$, $\nu > 1$, $\sigma \geq 1$. For small ω, the target steady state is locally stable under constant gain learning under the rule (29).*
(ii) Assume full price flexibility ($\gamma = 0$) and $\psi_p > \max[\frac{\pi^*(\omega/w_c)(1-w_c)}{(1-\beta)\beta}, \beta^{-1}\pi^*]$. For small ω, the target steady state is locally stable under constant gain learning under the rule (29).

When there is full price flexibility, however, the stability conditions depend on w_c and the ratio of ω to w_c, and the situation can ultimately be more stringent than the preceding proposition indicates. If ω and w_c are

[19] Budianto, Nakato, and Schmidt (2020) study AIT with exponential MA in a model with rational expectations.

both very small and $\omega \approx w_c$, then the condition for stability is far more demanding than the Taylor Principle. Eusepi and Preston (2018), section 4 study a related model that assumes $\omega = w_c$ and obtain similar results.

The fact that stability may depend on the private sector gain parameter suggests that the exponential moving average formulation of average inflation targeting can be a risky alternative to the weighted average formulation discussed in Section 4.1. For reasons of space we exclude this specification from further analysis.

4.3 Stability in Calibrated Model

Stability properties of the basic AIT rule (20) and the AIT rule with exponentially declining weights (28) are now examined further using a calibrated version of the model.

In the literature suggested calibrations for price adjustment costs in the NK model vary a great deal as they depend on estimates of frequency of price adjustment and markup and there are different estimates for both. For recent discussion see Honkapohja and Mitra (2020) who use the alternative values $\gamma = 42$, 128.21 or 350 for price adjustment cost parameter. For brevity, only a single standard calibration is adopted for other parameters. For a quarterly framework we set $\pi^* = 1.005$, $\beta = 0.99$, $\alpha = 0.7$, $\nu = 21$, $\sigma = \varepsilon = 1$, and $g = 0.2$. Policy parameters for the AIT rule are set at $\psi_p = 1.2$, $\psi_y = 1$ and $L = 6$ (i.e. five lags).

For the three calibrations of γ we compute the (approximate) least upper bound for the gain parameter ω_0, so that values $\omega > \omega_0$ lead to instability of the target steady state in the calibrated model. The results are as follows:

γ	42	128.21	350
ω_0 ($\mu = 1$)	0.00597	0.00986	0.02006
ω_0 ($\mu = .9$)	0.00684	0.01128	0.02301
ω_0 ($\mu = .7$)	0.00998	0.01643	0.03369
ω_0 ($\mu = .5$)	0.01743	0.02868	0.03889

Note: the first row gives the results with no discounting

Table 1: Least upper bounds ω_0 for instability

It is seen that discounting old data in the AIT rule contributes robustness of stability but a significant degree of discounting is needed.

These results can be compared to values for the gain parameter used in other calibrated and empirical macro models with adaptive learning.[20] There is no agreed range for gain parameters but the range could be something like $[0.002, 0.04]$, where smaller values are used in models of infinite horizon (IH) learning.[21]

The results in Table 1 can also be compared to least upper bound ω_0 for convergence with inflation (IT) and price level targeting (PLT). Table 2 makes the comparison.

γ	42	128.21	350
ω_0 (IT)	0.02935	0.03679	0.04172
ω_0 (PLT)	0.00805	0.00590	0.00413
ω_0 (AIT)	0.00404	0.00581	0.00809

Table 2: Least upper bounds in AIT, IT and PLT

It is seen that inflation targeting is clearly more robust than AIT or PLT. The latter two are fairly similar in terms of robustness.[22]

Looking at the results so far it is apparent that instability and lack of robustness under AIT monetary policy with opacity can be a major concern, though some mitigation of the problem is achieved by using discounting of older data when computing the average inflation target.

5 Learning When Policy Structure is Known

The analysis so far has focused on the consequences of opacity about policy, so that private agents do not know any details of the interest rate rule (20). Agents forecast using a misspecified time series model that does not nest the true structure of the economy which, apart from the policy rule, is forward-looking and does not have any lags of endogenous variables. Above it was

[20] For the AIT results in Table 2 we assume the interest rate rule is set according to (20). The AIT results in Table 2 and the $\mu = 1$ results in Table 1 are slightly different because (28) includes the normalizing constant: $(1-\mu)/(1-\mu^L) = 1/(1+\mu+...+\mu^{L-1})$.

[21] For empirical papers see e.g. Orphanides and Williams (2005), Branch and Evans (2006), Milani (2007) and the discussion in Section 4.2 in Eusepi, Giannoni, and Preston (2018).

[22] It may be recalled that according to Honkapohja and Mitra (2020) performance of PLT is much improved if private agents use more information about the policy framework.

shown that under opacity the economy may encounter instability under private agents' learning. Therefore it is important to consider situations where agents are more informed about aspects of the monetary policy framework.

5.1 Stability of the Target Steady State

Opacity is now replaced with the assumption that agents' forecasting model is based on the correct functional form of the stochastic process of endogenous variables, so that the agents' PLM has the same functional form as the economy.[23] Nevertheless, knowledge is imperfect as agents do not know the values of the structural parameters. More specifically, as concerns policy the agents know L but not the values of the policy coefficients ψ_p and ψ_y. The PLM or forecasting model thus has the $VAR(L-1)$ form.

For the analysis it is necessary to compute the general form of agents' linearized infinite-horizon (IH) decision rules. If agents are identical, the linearized IH Phillips curve obtained in Appendix A.4 is

$$\hat{\pi}_t = \kappa \hat{y}_t + \kappa \sum_{j=1}^{\infty} \beta^j \hat{y}^e_{t+j} \tag{34}$$

where $\hat{x}$ denotes a linearized variable, and κ is a complicated function of deep structural parameters. In Appendix A.4 it is shown that the linearized consumption function takes the form

$$\hat{y}_t = -\frac{c^*\beta}{\sigma\pi^*}\hat{R}_t + \sum_{j=1}^{\infty} \beta^j \left(\frac{1-\beta}{\beta}\hat{y}^e_{t+j} - \frac{c^*}{\sigma}\left(\beta \hat{R}^e_{t+j}/\pi^* - \hat{\pi}^e_{t+j}/(\beta\pi^*)\right)\right). \tag{35}$$

The linearized expression for the interest rate rule (28) outside the ZLB region is

$$\hat{R}_t = \psi_p \frac{1-\mu}{1-\mu^L} \sum_{k=0}^{L-1} \mu^k \hat{\pi}_{t-k}/\pi^* + \phi_y \hat{y}_t/y^*, \tag{36}$$

where $0 < \mu < 1$.[24] When needed, the rule (36) is modified to have the ZLB constraint.

Define $\hat{X} = (\hat{\pi}, \hat{y}, \hat{R})$. The temporary equilibrium $\hat{X}_t$ is given by (34), (35), and (36) given expectations $\hat{X}^e_{t+i}$. Stacking the model into first order

[23] More precisely, agents' PLM has the same functional form as the minimal state variable (MSV) solution of the linearized model.

[24] Note that $\frac{1-\mu}{1-\mu^L} = \frac{1}{\sum_{k=0}^{L-1}\mu^k}$.

form and using the general formulation developed in Appendix B, the actual law of motion (ALM) for the economy is

$$\tilde{X}_t = \tilde{K} + \textstyle\sum_{i=1}^{\infty} \beta^i \tilde{M}\ \tilde{X}^e_{t,t+i} + \tilde{N}\tilde{X}_{t-1},$$

and the PLM is of the form

$$\tilde{X}_t = \tilde{A}_0 + \tilde{A}\tilde{X}_{t-1}, \text{ where } \tilde{X}_t = (\hat{X}_t, ..., \hat{X}_{t-(L-2)})^T.$$

The mapping $PLM \rightarrow ALM$ can be simplified to

$$\begin{array}{lll} \tilde{A} & \rightarrow & \sum_{i=1}^{\infty} \beta^i \tilde{M}\tilde{A}^{i+1} + \tilde{N} \\ \tilde{A}_0 & \rightarrow & \tilde{K} + \sum_{i=1}^{\infty} \beta^i \tilde{M}(I + \tilde{A} + ... + \tilde{A}^i)\tilde{A}_0. \end{array}$$

E-stability analysis and stability conditions are given in Appendix B. Here our interest is in robustness of stability under learning when agents' PLM is assumed to match the MSV functional form of temporary equilibrium as there is more information about the AIT policy framework. To keep things tractable, assume that agents know the VAR coefficients in the REE law of motion $\bar{A}$.[25] Agents' PLM is:

$$\hat{X}_t = \bar{A}\hat{X}_{t-1} + \tilde{A}_{0,t} \tag{37}$$

where

$$\tilde{A}_{0,t} = \omega\left(\hat{X}_{t-1} - \bar{A}\hat{X}_{t-2} - \tilde{A}_{0,t-1}\right) + \tilde{A}_{0,t-1}.$$

Given the mapping from PLM to ALM, (63)-(64) in the Appendix B, the actual law of motion for x_t is

$$\begin{aligned} \hat{X}_t &= \bar{A}\hat{X}_{t-1} + \tilde{K} + \tilde{M}\left(I - \bar{A}\right)^{-1}\left(\frac{\beta}{1-\beta}I - \beta\bar{A}^2\left(I - \beta\bar{A}\right)^{-1}\right)\tilde{A}_{0,t} \\ &= \bar{A}\hat{X}_{t-1} + \tilde{K} + DT(\tilde{A}_0)\tilde{A}_{0,t} \end{aligned}$$

This implies

$$\hat{X}_{t-1} - \bar{A}\hat{X}_{t-2} = \tilde{K} + DT(\tilde{A}_0)\tilde{A}_{0,t-1},$$

[25] In this exercise, we numerically verify that the REE is E-stable, i.e., agents can learn $\bar{A}$. We assume $\bar{A}$ is known so that we may express the law of motion for agents' beliefs as a VAR, as shown in the text, and then appeal to standard linear difference equations techniques to identify the highest gain parameter (ω_0) above which instability under constant gain learning obtains. Note that ω_0 is likely lower if agents do not know $\bar{A}$ and must learn it over time.

and so

$$\begin{aligned}\tilde{A}_{0,t} &= \omega\left(\tilde{K} + DT(\tilde{A}_0)\tilde{A}_{0,t-1} - \tilde{A}_{0,t-1}\right) + \tilde{A}_{0,t-1} \\ &= \left(\omega DT(\tilde{A}_0)) + (1-\omega)I\right)\tilde{A}_{0,t-1} + \omega\tilde{K}.\end{aligned}$$

Therefore, $\tilde{A}_{0,t} \to \bar{A}_0$ if and only if the eigenvalues of

$$\omega DT(\tilde{A}_0)) + (1-\omega)I$$

are inside the unit circle.

This formulation is now applied to assess robustness of learning. The calibrated model is the same as in Section 4.3. For the three calibrations of γ the (approximate) least upper bound for the gain parameter ω_0, such that values $\omega > \omega_0$ lead to instability of the target steady state in the calibrated model is computed numerically. The results are as follows:

γ	42	128.21	350
ω_0 ($\mu = 1$)	0.02811	0.03431	0.03954
ω_0 ($\mu = .9$)	0.02826	0.03458	0.03979
ω_0 ($\mu = .5$)	0.02895	0.03593	0.04100

Table 3: Least upper bounds ω_0 for instability for AIT under structural information

Comparing Tables 1 and 3 it is evident that in an AIT policy framework learning with correct functional form is quite a bit more robust than learning under opacity about policy. The values of ω_0 are close to the highest values of the gain parameter that are used in empirical models with adaptive learning (see Section 4.3).

5.2 Escape from the ZLB Regime with AIT?

One reason for introducing AIT policy has been its potential in providing a framework that facilitates return from the regime of very low interest rates to a normal regime with the economy operating near the inflation target equilibrium.

At the outset it can be noted that AIT under opacity is not a mechanism for escape from the ZLB. Formally, if the economy is at or very near the low steady state $\pi_L < \pi^*$, $y_L < y^*$ and $R = 1$, dynamics with learning under the AIT interest rate rule (20) may not yield convergence back to the target π^*, y^*. If the interest rate is at zero ($R = 1$), then the dynamics are the same as those with inflation targeting, so the corresponding earlier result for IT monetary policy can be applied.[26]

Will this result change if monetary policy uses AIT with known general structure, called AIT(gs)? We consider this issue using the nonlinear NK model with (20) and agents' PLM taking the form (37) but with general coefficients

$$X_t = \tilde{A}_t X_{t-1} + \tilde{A}_{0,t}.$$

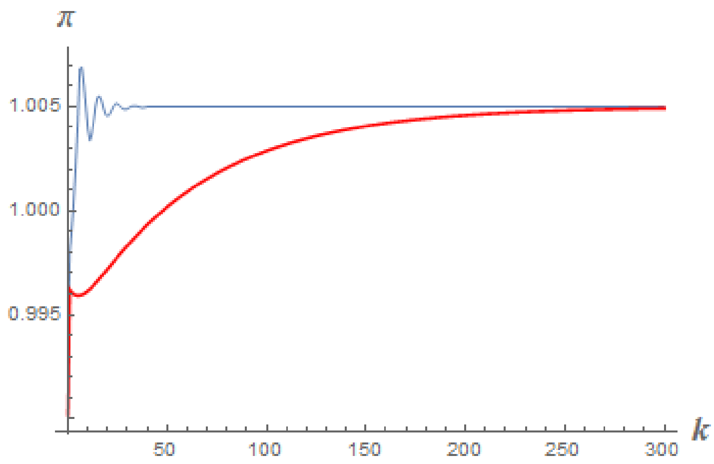
As a first example consider the case where the economy is very near the low steady state such that $\pi_0^e = \pi(0) = \pi(-1) = \ldots = \pi(-L+1) \approx \pi_L$, $y_0^e = y_0 \approx y_L$, $R_0 = R_0^e \approx 1$[27]. We assume $\tilde{A}_0$ is a zero matrix and $\tilde{A}_{0,0}$ is set in accordance with R_0^e, π_0^e, and y_0^e. Our assumption about $\tilde{A}_0$ may be a reasonable description of beliefs at the beginning of a transition from a standard IT policy regime to a well-communicated AIT regime. The basic calibration is the same as earlier in Section 4.3 with $\gamma = 128.21$ and gain equal to 0.005. Policy parameters are set as $\psi_p = 1.2$ and $\psi_y = 1$. We select $L = 6$, so that the AIT policy computes the average using a five-quarter window of past inflation data. The economy escapes the liquidity trap in this case, as shown by the blue curves in Figure 2 A-C.

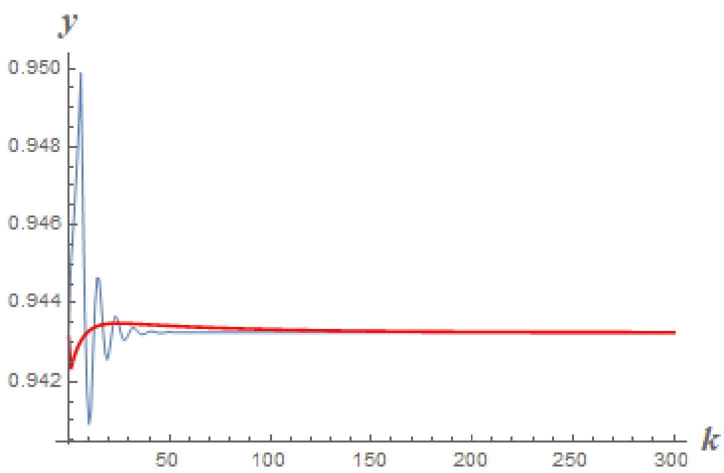
AIT(gs) is also compared with IT policy frameworks in Figure 2 A-C. The dynamics under IT are shown in red. The time paths of AIT(gs) are more oscillatory than paths under IT. Further, the dynamic properties of the economy under AIT(gs) and AIT opacity are sensitive to the value of the gain parameter. For example, we do not observe convergence to target steady state under AIT with opacity for gain values at 0.01 or higher, even though we observe convergence to target steady state under IT for gain values that exceed 0.06 (see Table 4). AIT(gs) is somewhat more robust than AIT with opacity but it can lead to divergence for high values of the gain. Speed of convergence of IT policy vs. AIT(gs) depends on the gain parameter. AIT(gs) converges faster if the gain value is small, but for higher gain values IT induces a much faster convergence than AIT(gs) and without generating

[26] See e.g. Evans, Guse, and Honkapohja (2008) and Benhabib, Evans, and Honkapohja (2014).

[27] In fact to facilitate the numerics we set R_0 and R slightly above 1.

large swings in inflation and output and recurring ZLB events.[28] It is thus clear that AIT does not necessarily outperform IT when the economy is initially near the low steady state.





[28]These results are available on request.

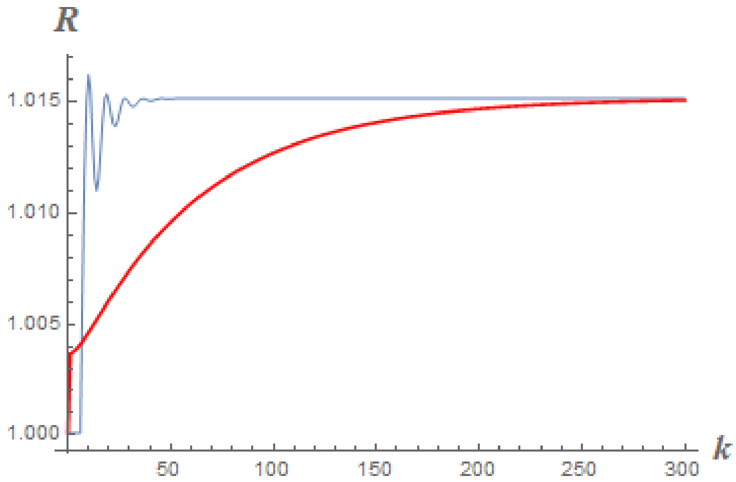


Figure 2 A-C: Escape of inflation, output and interest rate from liquidity trap under AIT with gs (blue) and IT (red).

It's well known that deflationary spirals occur under IT if expected inflation and output are somewhat below their low steady state values (see e.g. Benhabib, Evans, and Honkapohja (2014)). Will deflationary spirals also occur under AIT(gs) if expected inflation and output are significantly below the low steady state? The **domain of escape**[29] from the liquidity trap for different initial conditions $\pi_0^e \approx \pi(0) = \pi(-1) = \ldots = \pi(-L+1)$, $y_0^e \approx y_0$, and $R_0 = R_0^e \approx 1$, with $L = 6$ and $\omega = .002$ is shown below in Figure 3.[30]

[29] Domain of escape from the liquidity trap consists of points near the low steady state that lead to convergence to target steady state. It is part of domain of attraction of (π^*, y^*).

[30] See Appendix B.1 for the domain of escape under IT.

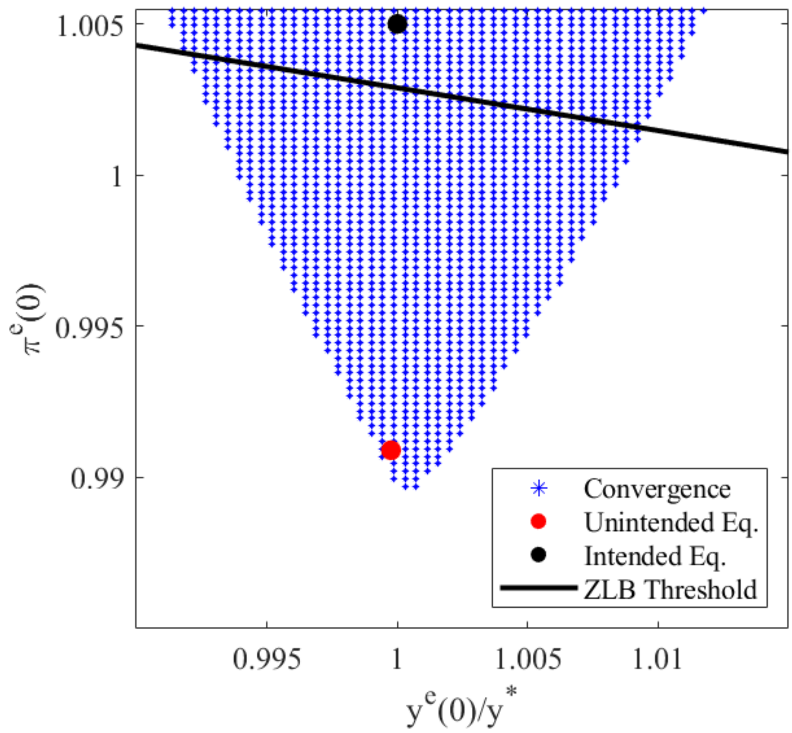


Figure 3: Domain of escape to target steady state

It is seen that there is a domain of escape from the liquidity trap, but it covers only a small area around the low steady state. In particular, if y_0^e is below y_L and π_0^e is approximately at level π_L, the economy does not escape from the liquidity trap.[31] By this measure both AIT and IT are less robust than PLT under similar information settings (see Honkapohja and Mitra (2020) for the corresponding results under PLT).

Robustness of the result about escape from ZLB with AIT(gs) was also studied. Five calibrations of the data window ($L = 1, 4, 6, 12, 20$) and two calibrations of ψ_p for each L ($\psi_p = 1.2$ and $\psi_p = 1.2/L$) were considered. For each calibration of L and ψ_p we computed an upper bound for the gain parameter ω_0, such that values $\omega > \omega_0$ lead to deflationary spirals under learning when actual and expected inflation and output are initially near the unintended steady state. For $\omega < \omega_0$ the economy eventually escapes the

[31]The figure also includes a line indicating the boundary of the ZLB region.

ZLB and returns to the intended steady state equilibrium, provided agents have correctly specified PLM and initialize the estimates of the PLM parameters as in the description of Figure 2. The results were not sensitive with respect to initial conditions for past inflation and initial values of the PLM parameters.[32]

It is seen that small values of L and ψ_p contribute robustness of the possibility of escape. In particular, we observe escape from the ZLB under IT ($L = 1$) for higher values of the gain parameter than under AIT ($L > 1$). We conclude that the performance of AIT policy in the nonlinear model with the ZLB is quite sensitive to the speed of learning, just as the success of AIT near the target steady state hinges on the magnitude of the gain parameter. Further, AIT does not clearly outperform IT when expectations are near the low steady state.

L	1 (IT)	4	6	12	20
ω_0 ($\psi_p = 1.2$)	0.06495	0.02175	0.01262	0.00582	0.00349
ω_0 ($\psi_p = 1.2/L$)	0.06495	0.03519	0.02107	0.00950	0.00549

Table 4: Critical lower bounds ω_0 for instability

6 Conclusion

Recent monetary policy challenges sparked interest in alternative policy frameworks, including AIT. The Federal Reserve adopted an AIT framework in 2020, but it did not communicate details about the structure of the policy, including the extent to which policy is history-dependent. This paper explored some implications of imperfect knowledge in an average inflation targeting regime with significant history dependence. For reasons of space, attention was limited to AIT interest rate rules that are symmetric (apart from the ZLB constraint).

Our results suggest that policymakers should be cautious when implementing AIT. An AIT policy practiced under opacity of its details can fail to anchor expectations around the target steady state if prices are flexible or the speed of learning is anything but very slow. Moreover, an AIT policy practiced under opacity will typically fail to instigate an escape from a liquidity trap.

[32] Detailed results are available on request.

AIT is, however, more robust if agents know that the current policy stance depends on a specific number of lags of the inflation rate. If agents incorporate information about the history-dependence of policy into learning, then the target steady state is fairly robustly stable, and AIT can even succeed in guiding the economy out of a liquidity trap. Policymakers may also mitigate the potential for instability under AIT by discounting past inflation data in their measure of the average inflation target.

There is plenty of room for future research. As a starting point for our analysis, we assume AIT is either conducted under opacity, or in an environment in which agents fully incorporate knowledge of the structure of policy into learning. Future work should examine whether AIT can ensure a locally stable target steady state, or initiate an escape from the liquidity trap, when communication from the central bank is imperfectly credible. As was already mentioned, performance of asymmetric rules, including switching rules under imperfect knowledge is another area worth exploring. We also focused on simple policy rules as a natural point of departure, though imperfect knowledge may have implications for optimal policy that have not yet been explored.

Appendices

A The Model

The objective for agent s is to maximize expected, discounted isoelastic cum quadratic utility subject to a standard flow budget constraint (in real terms) over the infinite horizon. The utility function for each period is standard except there is disutility from changing prices and no utility of real balances is displayed because of the cashless limit.

$$Max\ E_{0,s} \sum_{t=0}^{\infty} \beta^t \left[\frac{c_{t,s}^{1-\sigma}}{1-\sigma} - \frac{h_{t,s}^{1+\varepsilon}}{1+\varepsilon} - \frac{\gamma}{2} \left(\frac{P_{t,s}}{P_{t-1,s}} - 1 \right)^2 \right] \tag{38}$$

$$st.\ c_{t,s} + b_{t,s} + \Upsilon_{t,s} = R_{t-1} \pi_t^{-1} b_{t-1,s} + \frac{P_{t,s}}{P_t} y_{t,s}.$$

The final term in the utility function parameterizes the cost of adjusting prices in the spirit of Rotemberg (1982). The Rotemberg formulation is used rather than the Calvo (1983) model of price stickiness because it enables us to study global dynamics in the nonlinear system. The household decision problem is also subject to the usual "no Ponzi game" (NPG) condition. In (38) the expectations $E_{0,s}(.)$ are in general subjective and may not be rational.

Production function for good s is standard

$$y_{t,s} = h_{t,s}^{\alpha}, \text{ where } 0 < \alpha < 1.$$

There is no capital. Output is differentiated and firms operate under monopolistic competition. Each firm faces a downward-sloping demand curve

$$P_{t,s} = \left(\frac{y_{t,s}}{y_t}\right)^{-1/\nu} P_t. \tag{39}$$

Here $P_{t,s}$ is the profit maximizing price set by firm s consistent with its production $y_{t,s}$. The parameter ν is the elasticity of substitution between two goods and is assumed to be greater than one. y_t is aggregate output, which is exogenous to the firm.

The market clearing condition is

$$c_t + g_t = y_t.$$

The government consumes amount g_t of the aggregate good, collects the real lump-sum tax Υ_t from each consumer and issues bonds b_t to cover financing needs. Fiscal policy is assumed to follow a linear tax rule for lump-sum taxes $\Upsilon_t = \kappa_0 + \kappa b_{t-1}$, where $\beta^{-1} - 1 < \kappa < 1$, so fiscal policy is "passive" using terminology of Leeper (1991). Government purchases g_t is taken to be stochastic, so that $g_t = \bar{g} + \tilde{g}_t$, where the random part $\tilde{g}_t$ is an observable exogenous AR process

$$\tilde{g}_t = \rho \tilde{g}_{t-1} + v_t \tag{40}$$

with zero mean.[33]

A.1 Private sector optimization

With inclusion of the utility of real balances to the utility function in each period, the utility function household-producer s

$$U_{t,s} = \frac{c_{t,s}^{1-\sigma_1}}{1-\sigma_1} + \frac{\chi}{1-\sigma_2}\left(\frac{M_{t-1,s}}{P_t}\right)^{1-\sigma_2} - \frac{h_{t,s}^{1+\varepsilon}}{1+\varepsilon} - \frac{\gamma}{2}\left(\frac{P_{t,s}}{P_{t-1,s}} - 1\right)^2 \tag{41}$$

[33] For simplicity, it is assumed ρ is known (if not it could be estimated during learning). Only one shock is introduced in order to have a simple exposition.

and the constraints

$$
\begin{aligned}
st.\ c_{t,s}+b_{t,s}+\Upsilon_{t,s} &= R_{t-1}\pi_t^{-1}b_{t-1,s}+\frac{P_{t,s}}{P_t}y_{t,s}, \\
h_{t,s} &= y_{t,s}^{1/\alpha}.
\end{aligned}
$$

We compute the derivatives with respect to $(t-1)$-dated variables

$$
\begin{aligned}
\frac{\partial U_{t,s}}{\partial m_{t-1,s}} &= c_{t,s}^{-\sigma_1}\pi_t^{-1}+\chi(m_{t-1,s}\pi_t^{-1})^{-\sigma_2}, \\
\frac{\partial U_{t,s}}{\partial b_{t-1,s}} &= c_{t,s}^{-\sigma_1}R_{t-1}\pi_t^{-1},
\end{aligned}
$$

and with respect to t-dated variables

$$
\begin{aligned}
\frac{\partial U_{t,s}}{\partial m_{t,s}} &= \frac{\partial U_{t,s}}{\partial b_{t,s}} = -c_{t,s}^{-\sigma_1}, \\
\frac{\partial U_{t,s}}{\partial P_{t,s}} &= c_{t,s}^{-\sigma_1}Y_t(1-\upsilon)\left(\frac{P_{t,s}}{P_t}\right)^{-\upsilon}\frac{1}{P_t}+\frac{\upsilon}{\alpha}h_{t,s}^{1+\varepsilon}\frac{1}{P_{t,s}}.
\end{aligned}
$$

The Euler equations are

$$
\begin{aligned}
\frac{\partial U_{t,s}}{\partial m_{t,s}}+\beta E_{t,s}\frac{\partial U_{t+1,s}}{\partial m_{t,s}} &= 0, \\
\frac{\partial U_{t,s}}{\partial b_{t,s}}+\beta E_{t,s}\frac{\partial U_{t+1,s}}{\partial b_{t,s}} &= 0, \\
\frac{\partial U_{t,s}}{\partial P_{t,s}}+\beta E_{t,s}\frac{\partial U_{t+1,s}}{\partial P_{t,s}} &= 0.
\end{aligned}
$$

The second equation is just the consumption Euler equation, while combining the first and second equations yields the money demand function. The third equation is the condition for optimal price setting .

Applying the above conditions, in period t each household s is assumed to maximize its anticipated utility (41) under given expectations. As in Evans, Guse, and Honkapohja (2008), the first-order conditions for an optimum yield

$$
\begin{aligned}
0 &= -h_{t,s}^{\varepsilon}+\frac{\alpha\gamma}{\nu}(\pi_{t,s}-1)\pi_{t,s}\frac{1}{h_{t,s}} \\
&\quad +\alpha\left(1-\frac{1}{\nu}\right)y_t^{1/\nu}\frac{y_{t,s}^{(1-1/\nu)}}{h_{t,s}}c_{t,s}^{-\sigma}-\frac{\alpha\gamma\beta}{\nu}\frac{1}{h_{t,s}}E_{t,s}(\pi_{t+1,s}-1)\pi_{t+1,s},
\end{aligned}
\tag{42}
$$

$$c_{t,s}^{-\sigma} = \beta R_t E_{t,s}\left(\pi_{t+1}^{-1} c_{t+1,s}^{-\sigma}\right), \tag{43}$$

where $\pi_{t+1,s} = P_{t+1,s}/P_{t,s}$ and $E_{t,s}(.)$ denotes the (not necessarily rational) expectations of agents s formed in period t.

Equation (42) is one form of the nonlinear New Keynesian Phillips curve describing the optimal price-setting by firms. The term $(\pi_{t,s} - 1)\,\pi_{t,s}$ arises from the quadratic form of the adjustment costs, and this expression is increasing in $\pi_{t,s}$ over the allowable range $\pi_{t,s} \geq 1/2$. Equation (43) is the standard Euler equation giving the intertemporal first-order condition for the consumption path.

We now write the decision rules for consumption and inflation so that they depend on forecasts of key variables over the infinite horizon (IH).

A.2 The Infinite-horizon Phillips Curve

Starting with (42), let

$$Q_{t,s} = (\pi_{t,s} - 1)\,\pi_{t,s}. \tag{44}$$

The appropriate root for given Q is $\pi \geq \frac{1}{2}$ and so $Q \geq -\frac{1}{4}$ must be imposed to have a meaningful model. Using the production function $h_{t,s} = y_{t,s}^{1/\alpha}$ one can rewrite (42) as

$$Q_{t,s} = \frac{\nu}{\alpha\gamma} y_{t,s}^{(1+\varepsilon)/\alpha} - \frac{\nu - 1}{\gamma} y_t^{1/\nu} y_{t,s}^{(\nu-1)/\nu} c_{t,s}^{-\sigma} + \beta E_{t,s} Q_{t+1,s}, \tag{45}$$

and using the demand curve $y_{t,s}/y_t = (P_{t,s}/P_t)^{-\nu}$ gives

$$Q_{t,s} = \frac{\nu}{\alpha\gamma}(P_{t,s}/P_t)^{-(1+\varepsilon)\nu/\alpha} y_t^{(1+\varepsilon)/\alpha} - \frac{\nu - 1}{\gamma} y_t (P_{t,s}/P_t)^{-(\nu-1)} c_{t,s}^{-\sigma} + \beta E_{t,s} Q_{t+1,s}.$$

Defining

$$x_{t,s} \equiv \frac{\nu}{\alpha\gamma}(P_{t,s}/P_t)^{-(1+\varepsilon)\nu/\alpha} Y_t^{(1+\varepsilon)/\alpha} - \frac{\nu - 1}{\gamma} y_t (P_{t,s}/P_t)^{-(\nu-1)} c_{t,s}^{-\sigma} \tag{46}$$

and iterating the Euler equation[34] yields

$$Q_{t,s} = x_{t,s} + \sum_{j=1}^{\infty} \beta^j E_{t,s} x_{t+j,s}, \tag{47}$$

[34] It is assumed that expectations satisfy the law of iterated expectations.

provided that the transversality condition

$$\beta^j E_{t,s} x_{t+j,s} \to 0 \text{ as } j \to \infty \tag{48}$$

holds. It can be shown that (48) is an implication of the necessary transversality condition for optimal price setting. For further details see Benhabib, Evans, and Honkapohja (2014).

The variable $x_{t+j,s}$ is a mixture of aggregate variables and the agent's own future decisions. Thus it provides only a "conditional decision rule".[35] This equation for $Q_{t,s}$ can be the basis for decision-making as follows. So far only the agent's price-setting Euler equation and the above limiting condition (48) have been used. Some further assumptions are now made.

Agents are assumed to have point expectations, so that their decisions depend only on the mean of their subjective forecasts. The model outlined above stipulates that all agents s have the same utility and production functions. Initial money and debt holdings, and prices are assumed to be identical.

The assumption of representative agents includes private agents' forecasting, so that the agents have homogenous forecasts of the relevant variables. Thus all agents make the same decisions at each point in time. It is also assumed that from the past agents have learned the market clearing relation in temporary equilibrium, i.e. $c_{t,s} = y_t - g_t$ in per capita terms and thus agents impose in their forecasts that $c^e_{t+j} = y^e_{t,t+j} - g^e_{t,t+j}$, where $g^e_{t,t+j} = \bar{g} + \rho^j \tilde{g}_t$. In the case of constant fiscal policy this becomes $c^e_{t+j} = y^e_{t+j} - \bar{g}$.

The assumption of representative agents implies that in temporary equilibrium for all periods including the current one $P_{t,s} = P_{t,s'} = P_t$ for all agents s and s', see p. 224 in Benhabib, Evans, and Honkapohja (2014). In that paper it was additionally assumed that agents' expectations also satisfy $P^e_{t+j,s} = P^e_{t+j}$ for future periods $j = 1, 2, ...$ This assumption is not necessary and is adopted here purely as a simplification.[36]

[35] Conditional demand and supply functions are well known concepts in microeconomic theory.

[36] More extensive discussion of the generalization is available in Evans, Honkapohja, and Mitra (2020).

A.3 The Consumption Function

To derive the consumption function from (43), use the flow budget constraint and the NPG condition to obtain an intertemporal budget constraint.[37] Cashless limit is now assumed. First, define the asset wealth

$$a_t = b_t$$

as the holdings of real bonds and write the flow budget constraint as

$$a_t + c_t = y_t - \Upsilon_t + r_t a_{t-1}, \tag{49}$$

where $r_t = R_{t-1}/\pi_t$. Note that $(P_{jt}/P_t)y_{jt} = y_t$ is assumed, i.e. the representative agent assumption is invoked. Iterating (49) forward and imposing

$$\lim_{j\to\infty} (D^e_{t,t+j})^{-1} a^e_{t+j} = 0, \tag{50}$$

where

$$D^e_{t,t+j} = \frac{R_t}{\pi^e_{t+1}} \prod_{i=2}^{j} \frac{R^e_{t+i-1}}{\pi^e_{t+i}}$$

with $r^e_{t+i} = R^e_{t+i-1}/\pi^e_{t+i}$, one obtains the life-time budget constraint of the household

$$\begin{aligned} 0 &= r_t a_{t-1} + \Phi_t + \sum_{j=1}^{\infty} (D^e_{t,t+j})^{-1} \Phi^e_{t+j} && (51) \\ &= r_t a_{t-1} + \phi_t - c_t + \sum_{j=1}^{\infty} (D^e_{t,t+j})^{-1} (\phi^e_{t+j} - c^e_{t+j}), && (52) \end{aligned}$$

where

$$\begin{aligned} \Phi^e_{t+j} &= y^e_{t+j} - \Upsilon^e_{t+j} - c^e_{t+j}, \\ \phi^e_{t+j} &= \Phi^e_{t+j} + c^e_{t+j} = y^e_{t+j} - \Upsilon^e_{t+j}. \end{aligned} \tag{53}$$

Here all expectations are formed in period t, which is indicated in the notation for $D^e_{t,t+j}$ but is omitted from the other expectational variables.

Invoking the relations

$$c^e_{t+j} = (\beta^j D^e_{t,t+j})^{1/\sigma} c_t, \tag{54}$$

[37] Recall that the model is a cashless limit of the corresponding models in the cited earlier literature.

which are an implication of the consumption Euler equation (43), yields

$$c_t(1-\beta)^{-1} = r_t a_{t-1} + y_t - \Upsilon_t + \sum_{j=1}^{\infty} (D^e_{t,t+j})^{-1} \phi^e_{t+j} - \sum_{j=1}^{\infty} (D^e_{t,t+j})^{-1} (\beta^j D^e_{t,t+j})^{1/\sigma} c_t. \tag{55}$$

As we have $\phi^e_{t+j} = y^e_{t+j} - \Upsilon^e_{t+j}$, we have

$$c_t = \left(1 + \sum_{j=1}^{\infty} \beta^{j/\sigma} (D^e_{t,t+j})^{(1-\sigma)/\sigma}\right)^{-1} \left(r_t b_{t-1} + \sum_{j=0}^{\infty} (D^e_{t,t+j})^{-1} \phi^e_{t+j}\right).$$

So far it is not assumed that households act in a Ricardian way, i.e. they have not imposed the intertemporal budget constraint (IBC) of the government. To simplify the analysis, it is assumed that consumers are Ricardian, which allows to modify the consumption function as in Evans and Honkapohja (2010). See Evans, Honkapohja, and Mitra (2012) for discussion of the assumptions under which Ricardian Equivalence holds along a path of temporary equilibria with learning if agents have an infinite decision horizon. The government flow constraint is

$$b_t + \Upsilon_t = \bar{g} + \tilde{g}_t + r_t b_{t-1} \text{ or } b_t = \Delta_t + r_t b_{t-1} \text{ where } \Delta_t = \bar{g} + \tilde{g}_t - \Upsilon_t.$$

By forward substitution, and assuming

$$\lim_{T\to\infty} D_{t,t+T} b_{t+T} = 0, \tag{56}$$

one gets

$$0 = r_t b_{t-1} + \Delta_t + \sum_{j=1}^{\infty} D^{-1}_{t,t+j} \Delta_{t+j}. \tag{57}$$

Note that Δ_{t+j} is the primary government deficit in $t+j$, measured as government purchases less lump-sum taxes. Under the Ricardian assumption, agents at each time t expect this constraint to be satisfied, i.e.

$$\begin{aligned} 0 &= r_t b_{t-1} + \Delta_t + \sum_{j=1}^{\infty} (D^e_{t,t+j})^{-1} \Delta^e_{t+j}, \text{ where} \\ \Delta^e_{t+j} &= \bar{g} + \rho^j \tilde{g}_t - \Upsilon^e_{t+j} \text{ for } j = 1, 2, 3, \ldots. \end{aligned}$$

A Ricardian consumer assumes that (56) holds. His flow budget constraint (49) can then be written as:

$$b_t = r_t b_{t-1} + \psi_t, \text{ where } \psi_t = y_t - \Upsilon_t - c_t.$$

The relevant transversality condition is now (56). Iterating forward and using (54) together with (56) yields the consumption function (5) in the main text.

The aggregate demand function takes the form

$$y_t = g_t + \left(\sum_{j=1}^{\infty} \beta^{j/\sigma} (D^e_{t,t+j})^{(1-\sigma)/\sigma} \right)^{-1} \sum_{j=1}^{\infty} (D^e_{t,t+j})^{-1} (y^e_{t+j} - (\bar{g} + \rho^j \tilde{g}_t)), \quad (58)$$

where the discount factor is given by (16).

A.4 Linearized IH Behavioral Rules

Linearizing (13) and (47) around the intended steady state and rearranging gives the following linearized expression for the Phillips curve:

$$\hat{\pi}_t = \kappa \hat{y}_t + \kappa \sum_{j=1}^{\infty} \beta^j \hat{y}^e_{t+j}$$

where $\hat{x}$ denotes a linearized variable, and κ is a complicated function of deep structural parameters.

The consumption function (15) is linearized as follows. For the sake of brevity, assume $\tilde{g}_t = 0$. The discount factor $D^e_{t,t+j}$ has the linearization

$$\hat{D}^e_{t,t+j} = \beta^{1-j} \sum_{i=1}^{j} \left(\hat{R}^e_{t+i-1}/\pi^* - \hat{\pi}^e_{t+i}/(\beta\pi^*) \right).$$

Linearizing the left-hand-side of (15) gives

$$\begin{aligned}
& \frac{\beta}{1-\beta} \hat{c}_t + c^* \frac{1-\sigma}{\sigma} \sum_{j \geq 1} \beta^{j/\sigma} \left(\beta^{-j} \right)^{(1-\sigma)/\sigma - 1} \hat{D}^e_{t,t+j} \\
= \quad & \frac{\beta}{1-\beta} \hat{c}_t + c^* \frac{1-\sigma}{\sigma} \sum_{j \geq 1} \beta^{2j} \hat{D}^e_{t,t+j} \\
= \quad & \frac{\beta}{1-\beta} \hat{c}_t + c^* \frac{1-\sigma}{\sigma} \sum_{j \geq 1} \beta^{j+1} \sum_{i=1}^{j} \left(\hat{R}^e_{t+i-1}/\pi^* - \hat{\pi}^e_{t+i}/(\beta\pi^*) \right).
\end{aligned}$$

Linearizing the right-hand-side of (15) gives

$$\begin{aligned}
& \sum_{j \geq 1} \beta^j \hat{y}^e_{t+j} - c^* \sum_{j \geq 1} \beta^{2j} \hat{D}^e_{t,t+j} \\
= \quad & \sum_{j \geq 1} \beta^j \hat{y}^e_{t+j} - c^* \sum_{j \geq 1} \beta^{j+1} \sum_{i=1}^{j} \left(\hat{R}^e_{t+i-1}/\pi^* - \hat{\pi}^e_{t+i}/(\beta\pi^*) \right).
\end{aligned}$$

Equating the two sides of (15) and rearranging gives

$$\hat{y}_t = -\frac{c^*\beta}{\sigma\pi^*}\hat{R}_t + \sum_{j=1}^{\infty}\beta^j\left(\frac{1-\beta}{\beta}\hat{y}^e_{t+j} - \frac{c^*}{\sigma}\left(\beta\hat{R}^e_{t+j}/\pi^* - \hat{\pi}^e_{t+j}/(\beta\pi^*)\right)\right). \quad (59)$$

A.5 Formulation of Learning

The basic model apart from the AIT rule is purely forward-looking while the observable exogenous shock $\tilde{g}_t$ is an AR(1) process. Assuming opacity about AIT rule, then the appropriate PLM is a linear projection of $(y_{t+1}, \pi_{t+1}, R_{t+1})$ onto an intercept and the exogenous shock and agents estimate the regressions

$$s_u = a_s + b_s\tilde{g}_{u-1} + \varepsilon_{s,u},$$

where $s = y$, π, R by using a version of least squares and data for periods $u = 1, ..., t-1$. The latter is a common timing assumption in the learning literature; at the end of period $t-1$ the parameters are estimated using data through to period $t-1$. This gives estimates $a_{y,t-1}$, $b_{y,t-1}$, $a_{\pi,t-1}$, $b_{\pi,t-1}$, $a_{R,t-1}$, $b_{R,t-1}$ and using these estimates and data at time t the forecasts are given by

$$s^e_{t+j} = a_{s,t-1} + b_{s,t-1}\rho^j\tilde{g}_t,$$

for future periods $t+j$. These forecasts are then substituted into the system to determine a temporary equilibrium of the economy in periods $t+j$. With the new data point the estimates are updated and the process continues.

It turns out that the technical analysis of convergence and computation of domains of attraction can be carried out using a simplification. Apart from the unknown policy rule the model is purely forward-looking while $\tilde{g}_t$ is an AR(1) process. Under opacity the PLM is a linear projection of the state variables $(y_{t+1}, \pi_{t+1}, R_{t+1})$ onto an intercept and the exogenous shock and in this case convergence of learning to a fixed point is fully governed by the dynamics of intercepts.

Thus, stability of a steady state can be validly assessed using the simplifying assumption that $\tilde{g}_t$ is identically zero. The agents are thought to estimate the long run mean values of state variables, called "steady state learning". The latter is used here as a technical tool. In simulations of the stochastic model agents are assumed to do least squares learning.

A.6 Model with Flexible Prices

In the special case of the NK model with flexible prices there is no Phillips curve and the first order condition (42) is replaced by the static condition

$$\frac{\partial U_{t,s}}{\partial P_{t,s}} = c_{t,s}^{-\sigma_1} Y_t (1-\upsilon) \left(\frac{P_{t,s}}{P_t}\right)^{-\upsilon} \frac{1}{P_t} + \frac{\upsilon}{\alpha} h_{t,s}^{1+\varepsilon} \frac{1}{P_{t,s}} = 0.$$

Under symmetry it yields

$$c_t^{-\sigma_1} \alpha \frac{1-\upsilon}{\upsilon} + h_t^{1+\varepsilon-\alpha} = 0, \tag{60}$$

Steady-state learning with point expectations is formalized as before in Section 3.2. The temporary equilibrium equations with steady state learning are as follows.

1. With Ricardian consumers the market clearing equation is $y_t = g_t + c_t$, yields

$$y_t = \bar{g} + (1-\beta)\left[y_t - \bar{g} + (y_t^e - \bar{g})\left(\frac{\pi_t^e}{R_t}\right)\left(\frac{R_t^e}{R_t^e - \pi_t^e}\right)\right] \tag{61}$$

   as the aggregate demand relation.

2. A static labor-consumption optimality condition (60) can be combined with market clearing to obtain

$$y_t = \left(\alpha \frac{\upsilon - 1}{\upsilon} (y_t - g_t)^{-\sigma_1}\right)^{\alpha/(1+\varepsilon-\alpha)}. \tag{62}$$

   Looking at (62) it is evident that output in temporary equilibrium is exogenous.[38]

3. Interest rate rule (17) discussed below.

If one substitutes the interest rate rule (17) and also an exogenous value of output into (61), the model effectively says that the nominal interest rate R_t (and π_t via the policy rule) is the variable that establishes equality of aggregate demand and supply in temporary equilibrium. Using the interest rate rule (17) this yields the temporary equilibrium value for inflation π_t.

The system has three expectational variables: output y_t^e, inflation π_t^e, and interest rate R_t^e. The evolution of expectations is in accordance with steady state learning. Proposition 2 gives the instability result.

[38] Exogeneity of output holds in the classical monetary model, see e.g. Gali (2008), chapter 1.

B E-Stability for Linear Multivariate IH Models

Consider a linearized multivariate model in which agents are forward-looking with infinite horizon and there are $L-1$ lags of endogenous variables. Its general form is

$$\begin{aligned}\tilde{X}_t &= (X_t, ..., X_{t-(L-1)})^T \\ X_t &= K + \textstyle\sum_{i=1}^{\infty} M_i \, X^e_{t,t+i} + \sum_{j=1}^{L} N_j X_{t-j}.\end{aligned}$$

Stacking the system into first order form gives

$$\tilde{X}_t = \tilde{K} + \textstyle\sum_{i=1}^{\infty} \tilde{M}_i \, \tilde{X}^e_{t,t+i} + \tilde{N}\tilde{X}_{t-1}$$

which written out is

$$\begin{aligned}\begin{pmatrix} X_t \\ X_{t-1} \\ \vdots \\ X_{t-(L-1)} \end{pmatrix} &= \begin{pmatrix} K \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \textstyle\sum_{i=1}^{\infty} \begin{pmatrix} M_i & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} X^e_{t,t+i} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \\ &\quad + \begin{pmatrix} N_1 & N_2 & \cdots & N_L \\ I & 0 & \cdots & 0 \\ \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & I & 0 \end{pmatrix} \begin{pmatrix} X_{t-1} \\ X_{t-2} \\ \vdots \\ X_{t-L} \end{pmatrix},\end{aligned}$$

and so

$$\begin{aligned}\tilde{X}_t &= \begin{pmatrix} X_t \\ X_{t-1} \\ \vdots \\ X_{t-(L-1)} \end{pmatrix}, \, \tilde{M}_i = \begin{pmatrix} M_i & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix} \\ \tilde{X}^e_{t,t+i} &= \begin{pmatrix} X^e_{t,t+i} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \, \tilde{N} = \begin{pmatrix} N_1 & N_2 & \cdots & N_L \\ I & 0 & \cdots & 0 \\ \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & I & 0 \end{pmatrix}.\end{aligned}$$

The PLM is

$$X_t = A_0 + \textstyle\sum_{j=1}^{L} A_j X_{t-j}$$

so

$$\tilde{X}_t = \tilde{A}_0 + \tilde{A}\tilde{X}_{t-1},$$

where

$$\tilde{A} = \begin{pmatrix} A_1 & A_2 & \cdots & A_L \\ I & 0 & \cdots & 0 \\ \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & I & 0 \end{pmatrix} \text{ and } \tilde{A}_0 = \begin{pmatrix} A_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

The ALM is

$$\begin{aligned} \tilde{X}_t &= \tilde{K} + \textstyle\sum_{i=1}^{\infty} \tilde{M}_i[(I + \tilde{A} + ... + \tilde{A}^i)\tilde{A}_0 + \tilde{A}_{t-1}^{i+1}\tilde{X}_{t-1}] + \tilde{N}\tilde{X}_{t-1} \\ &= \tilde{K} + \textstyle\sum_{i=1}^{\infty} \tilde{M}_i(I + \tilde{A} + ... + \tilde{A}^i)\tilde{A}_0 + \textstyle\sum_{i=1}^{\infty} \tilde{M}_i\tilde{A}^{i+1}\tilde{X}_{t-1} + \tilde{N}\tilde{X}_{t-1} \end{aligned}$$

and the mapping $PLM \to ALM$ is

$$\begin{aligned} \tilde{A} &\to \textstyle\sum_{i=1}^{\infty} \tilde{M}_i\tilde{A}^{i+1} + \tilde{N} \\ \tilde{A}_0 &\to \tilde{K} + \textstyle\sum_{i=1}^{\infty} \tilde{M}_i(I + \tilde{A} + ... + \tilde{A}^i)\tilde{A}_0. \end{aligned}$$

In models with infinite decision horizons it is often the case that

$$\tilde{M}_i = \beta^i\tilde{M},$$

where β is the subjective discount factor. In this case the mapping $PLM \to ALM$ simplifies to

$$\begin{aligned} \tilde{A} &\to \textstyle\sum_{i=1}^{\infty} \beta^i\tilde{M}\tilde{A}^{i+1} + \tilde{N} \\ \tilde{A}_0 &\to \tilde{K} + \textstyle\sum_{i=1}^{\infty} \beta^i\tilde{M}(I + \tilde{A} + ... + \tilde{A}^i)\tilde{A}_0 \end{aligned}$$

or

$$\tilde{A} \to \tilde{M}\left(I - \beta\tilde{A}\right)^{-1}\beta\tilde{A}^2 + \tilde{N} \tag{63}$$

$$\tilde{A}_0 \to \tilde{K} + \tilde{M}\left(I - \tilde{A}\right)^{-1}\left(\frac{\beta}{1-\beta}I - \beta\tilde{A}^2\left(I - \beta\tilde{A}\right)^{-1}\right)\tilde{A}_0. \tag{64}$$

In this case it is straight-forward to obtain the E-stability conditions.

E-stability Conditions: Let $(\tilde{A}, \tilde{A}_0) = (\bar{A}, \bar{A}_0)$ denote a rational expectations equilibrium. The REE, $(\bar{A}, \bar{A}_0)$, is E-stable if the real parts of the eigenvalues of

$$
\begin{aligned}
DT(\tilde{A}) &= \left(\left(I-\beta\bar{A}\right)^{-1}\beta\bar{A}^2\right)^T \otimes \left(\tilde{M}\left(I-\beta\bar{A}\right)^{-1}\beta\right) + \\
&\quad I\otimes\left(\tilde{M}\left(I-\beta\bar{A}\right)^{-1}\beta\bar{A}\right) + \bar{A}^T\otimes\left(\tilde{M}\left(I-\beta\bar{A}\right)^{-1}\beta\right) \\
DT(\tilde{A}_0) &= \tilde{M}\left(I-\tilde{A}\right)^{-1}\left(\frac{\beta}{1-\beta}I-\beta\tilde{A}^2\left(I-\beta\tilde{A}\right)^{-1}\right)
\end{aligned}
$$

are less than one.

B.1 Domain of Escape for Inflation Targeting

The figure shows the domain of escape under IT. The basic parameter settings are as given earlier.

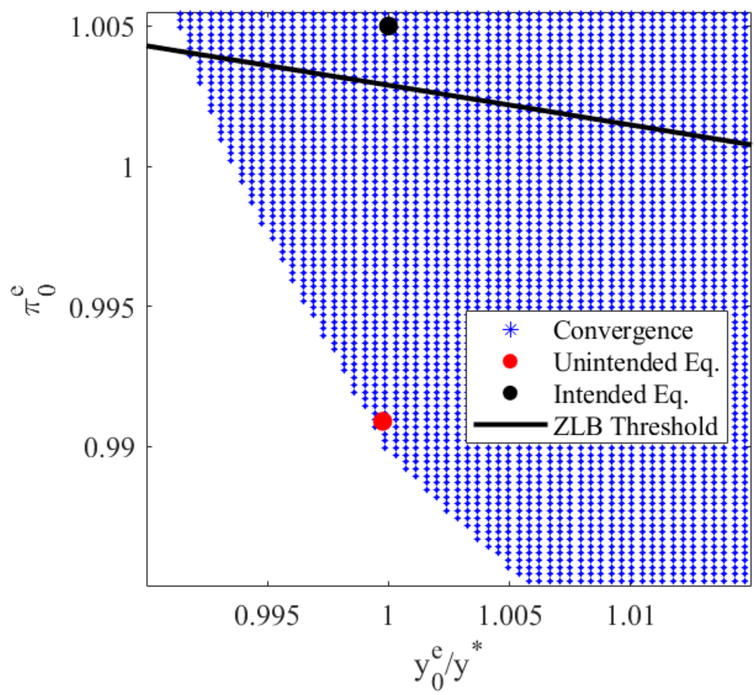


Figure A.1: Domain of escape for IT.

It should be noted that the result about escape from low steady state π_L, y_L differs from that in Figure 1 of Honkapohja and Mitra (2020). There are some differences in parameter values and most importantly in initial conditions for R_0 and R_0^e. In computing conditional domain of attraction it is natural to assume that R_0 and R_0^e are approximately equal to the steady state value R^*, whereas computation of domain of escape Figure A.1 assumes that R_0 and R_0^e are approximately 1.

C Proofs

C.1 Proof of Proposition 1

In the linearization (26)-(27) we get

$$DF_x = \begin{pmatrix} 1 & 0 & \frac{\beta(y^*-g)}{\pi^*\sigma} \\ -\kappa & 1 & 0 \\ -\frac{\psi_y}{y^*} & -\frac{\psi_p}{\pi^*} & 1 \end{pmatrix}$$

$$DF_{x^e} = \begin{pmatrix} -1 & \frac{-(g-y^*)}{\pi^*\sigma(\beta-1)} & \frac{\beta^2(g-y^*)}{\pi^*\sigma(\beta-1)} \\ \frac{\beta}{\beta-1}\kappa & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$DF_{x_{-i}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\psi_p/\pi^* & 0 \end{pmatrix}, \; i = 1, ..., L-1.$$

where

$$\kappa = \frac{\nu\left(\frac{(\nu-1)\sigma y^*(y^*-\bar{g})^{-\sigma-1}}{\nu} - \frac{(\nu-1)(y^*-\bar{g})^{-\sigma}}{\nu} + \frac{(\epsilon+1)y*^{\frac{\epsilon+1}{\alpha}-1}}{\alpha^2}\right)}{\gamma(2\pi^*-1)} \geq 0$$

if $\sigma > (y^* - \bar{g})/y^*$. It follows that

$$
\begin{aligned}
M &= -(DF_x)^{-1} DF_{x^e} = \\
&\begin{pmatrix}
\frac{y^*\left(\beta^2\kappa\psi_p(y^*-\bar{g})+(\beta-1)\pi*^2\sigma\right)}{a} & \frac{\pi^* y^*(\bar{g}-y^*)}{a} & \frac{\beta^2\pi^* y^*(y^*-\bar{g})}{a} \\
\frac{\kappa(-\pi^*)\left(\beta^2\psi_y(y^*-\bar{g})+\pi^*\sigma y^*\right)}{a} & \frac{\kappa\pi^* y^*(\bar{g}-y^*)}{a} & \frac{\beta^2\kappa\pi^* y^*(y^*-\bar{g})}{a} \\
\frac{\pi^*\sigma((\beta-1)\pi^*\psi_y-\kappa\psi_p y^*)}{a} & \frac{(\bar{g}-y^*)(\pi^*\psi_y+\kappa\psi_p y^*)}{a} & \frac{\beta^2(y^*-\bar{g})(\pi^*\psi_y+\kappa\psi_p y^*)}{a}
\end{pmatrix}, \\
N_i &= -(DF_x)^{-1} DF_{x_i} = \\
&\begin{pmatrix}
0 & \frac{\beta\psi_p y^*(\bar{g}-y^*)}{\beta\pi^*\psi_y(y^*-\bar{g})+\beta\kappa\psi_p y^*(y^*-\bar{g})+(\pi^*)^2\sigma y^*} & 0 \\
0 & \frac{\beta\kappa\psi_p y^*(\bar{g}-y^*)}{\beta\pi^*\psi_y(y^*-\bar{g})+\beta\kappa\psi_p y^*(y^*-\bar{g})+(\pi^*)^2\sigma y^*} & 0 \\
0 & \frac{\pi^*\sigma\psi_p y^*}{\beta\pi^*\psi_y(y^*-\bar{g})+\beta\kappa\psi_p y^*(y^*-\bar{g})+(\pi^*)^2\sigma y^*} & 0
\end{pmatrix}, \ i = 1, ..., L-1.
\end{aligned}
$$

where

$$a = (\beta - 1)\left(\beta\pi^*\psi_y(y^* - \bar{g}) + \beta\kappa\psi_p y^*(y^* - \bar{g}) + (\pi^*)^2\sigma y^*\right) < 0.$$

Introduce the notation $x_t = (y_t, \pi_t, R_t)$ etc. Modifying the system (25), (26)

$$Z_t = QZ_{t-1}, \text{ where} \tag{65}$$

$$Z_t = \begin{pmatrix} x_t^e & x_t & x_{t-1} & x_{t-2} & \cdots & x_{t-(L-2)} \end{pmatrix}^T$$

$$Q = \begin{pmatrix}
(1-\omega)I_3 & \omega I_3 & 0 & \cdots & 0 & 0 \\
(1-\omega)M & \omega M + N_1 & N_2 & \cdots & N_{L-2} & N_{L-1} \\
0 & I_3 & 0 & \cdots & 0 & 0 \\
0 & 0 & I_3 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & I_3 & 0
\end{pmatrix}.$$

For stability, the roots of $P(\lambda) = Det[Q - \lambda I_{3L}]$ must be inside the unit circle. One can show that

$$P(\lambda) = \lambda^{2L-2}(1 + \omega - \lambda)\tilde{P}(\lambda)$$

Thus, the roots of $P(\lambda)$ are inside the unit circle if and only if the roots of $\tilde{P}(\lambda)$ are inside the unit circle. In the limit $\omega \to 0$, we have

$$\tilde{P}(\lambda) = (1 - \lambda)^2(\lambda^{L-1} + h\sum_{k=0}^{L-2}\lambda^k)$$

where

$$h = \frac{\beta\kappa\psi_p y^*(y^* - \bar{g})}{\beta\pi^*\psi_y(y^* - \bar{g}) + \beta\kappa\psi_p y^*(y^* - \bar{g}) + (\pi^*)^2\sigma y^*} \in (0,1)$$

if $\gamma > 0$. Using the stability criterion in Jury (1961), the roots of $(\lambda^{L-1} + h\sum_{k=0}^{L-2}\lambda^k)$ are inside the unit circle if and only if[39]

$$1 - \frac{kh^2}{1+(k-1)h} > 0, k = 1, \ldots, L,$$

which is satisfied for all L. Therefore, the roots of $P(\lambda)$ are inside the unit circle if $\partial\lambda/\partial\omega < 0$ evaluated at $\omega = 0$ and $\lambda = 1$. To evaluate the derivative, we consider the Taylor series expansion of $\tilde{P}(\lambda)$ up to second order at point (λ_0, ω_0). Let $(d\lambda, d\omega) = (\lambda, \omega) - (\lambda_0, \omega_0)$. Then

$$\begin{aligned}\tilde{P}(\lambda,\omega) &= \tilde{P}(\lambda_0,\omega_0) + \tilde{P}_\lambda(\lambda_0,\omega_0)d\lambda + \tilde{P}_\omega(\lambda_0,\omega_0)d\omega + \\ &\quad \tilde{P}_{\lambda\lambda}(\lambda_0,\omega_0)\frac{d\lambda^2}{2} + \tilde{P}_{\lambda\omega}(\lambda_0,\omega_0)d\lambda d\omega + \tilde{P}_{\omega\omega}(\lambda_0,\omega_0)\frac{d\omega^2}{2} + Q,\end{aligned}$$

where subscripts denote partial derivatives and Q is a remainder.

Now

$$\begin{aligned}\tilde{P}_\omega(\lambda_0,\omega_0) &= 0 \\ \tilde{P}_\lambda(\lambda_0,\omega_0) &= 0\end{aligned}$$

so we get the approximation

$$\tilde{P}(\lambda,\omega) = \tilde{P}(\lambda_0,\omega_0) + \tilde{P}_{\lambda\lambda}(\lambda_0,\omega_0)\frac{d\lambda^2}{2} + \tilde{P}_{\lambda\omega}(\lambda_0,\omega_0)d\lambda d\omega + \tilde{P}_{\omega\omega}(\lambda_0,\omega_0)\frac{d\omega^2}{2}.$$

Now impose

$$\tilde{P}(\lambda,\omega) - \tilde{P}(\lambda_0,\omega_0) = 0$$

to compute the derivative of the implicit function. So we have

$$\tilde{P}_{\lambda\omega}(\lambda_0,\omega_0)d\lambda d\omega + \tilde{P}_{\omega\omega}(\lambda_0,\omega_0)\frac{d\omega^2}{2} + \tilde{P}_{\lambda\lambda}(\lambda_0,\omega_0)\frac{d\lambda^2}{2} = 0$$

or

$$\frac{d\lambda}{d\omega} = \frac{-1}{2}\left(\frac{\tilde{P}_{\omega\omega}(\lambda_0,\omega_0)}{\tilde{P}_{\lambda\omega}(\lambda_0,\omega_0)} + \frac{\tilde{P}_{\lambda\lambda}(\lambda_0,\omega_0)}{\tilde{P}_{\lambda\omega}(\lambda_0,\omega_0)}\left(\frac{d\lambda}{d\omega}\right)^2\right)$$

[39] Proof in Mathematica available on request.

Evaluating the partial derivatives at $(\lambda_0, \omega_0) = (1, 0)$ we have

$$
\begin{aligned}
\tilde{P}_{\omega\omega}(1,0) &= (-1)^L \frac{2(y^* - \bar{g})((1-\beta)\beta\pi^*\psi_y + \kappa y^*(L\beta\psi_p - \pi^*))}{(\beta-1)^2\left(\beta\pi^*\psi_y(y^*-\bar{g}) + \beta\kappa\psi_p y^*(y^*-\bar{g}) + (\pi^*)^2\sigma y^*\right)} \\
\tilde{P}_{\lambda\lambda}(1,0) &= (-1)^L \frac{2\left(\beta\pi^*\psi_y(y^*-\bar{g}) + L\beta\kappa\psi_p y^*(y^*-\bar{g}) + (\pi^*)^2\sigma y^*\right)}{\beta\pi^*\psi_y(y^*-\bar{g}) + \beta\kappa\psi_p y^*(y^*-\bar{g}) + (\pi^*)^2\sigma y^*} \\
\tilde{P}_{\lambda\omega}(1,0) &= (-1)^L \frac{\kappa y^*(y^*-\bar{g})(\pi^* - 2L\beta\psi_p) + (2-\beta)\beta\pi^*\psi_y(\bar{g}-y^*) - (1-\beta)(\pi^*)^2\sigma y^*}{(\beta-1)\left(\beta\pi^*\psi_y(y^*-\bar{g}) + \beta\kappa\psi_p y^*(y^*-\bar{g}) + (\pi^*)^2\sigma y^*\right)}
\end{aligned}
$$

One can show that $P_{\omega\omega}(1,0) > 0$, $P_{\lambda\lambda}(1,0) > 0$, $P_{\lambda\omega}(1,0) > 0$ if L is even and $P_{\omega\omega}(1,0) < 0$, $P_{\lambda\lambda}(1,0) < 0$, $P_{\lambda\omega}(1,0) < 0$ if L is odd. Therefore, $\partial\lambda/\partial\omega < 0$ and we have stability for small ω and $\kappa > 0$.

C.2 Proof of Proposition 2

In the case $\gamma = 0$ the dynamics of output expectations do not depend on the rest of the system and can be shown to be locally convergent. Introducing the notation $\tilde{x}_t = (\pi_t, R_t)^T$, the linearization (26)-(27) becomes

$$
\begin{aligned}
\tilde{M} &\equiv -(D\tilde{F}_x)^{-1} D\tilde{F}_{x^e} = \begin{pmatrix} \frac{\pi^*}{\psi_p\beta(1-\beta)} & -\frac{\beta\pi^*}{\psi_p(1-\beta)} \\ \frac{1}{\beta(1-\beta)} & -\frac{\beta}{(1-\beta)} \end{pmatrix} \text{ and} \\
\tilde{N}_i &\equiv -(D\tilde{F}_x)^{-1} D\tilde{F}_{x_i}. = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \; i = 1, ..., L-1.
\end{aligned}
$$

The system becomes

$$
\tilde{Z}_t = \tilde{Q}\tilde{Z}_{t-1}, \text{ where} \tag{66}
$$

$$
\tilde{Z}_t = \begin{pmatrix} \tilde{x}_t^e & \tilde{x}_t & \tilde{x}_{t-1} & \tilde{x}_{t-2} & \cdots & \tilde{x}_{t-(L-2)} \end{pmatrix}^T
$$

$$
\tilde{Q} = \begin{pmatrix}
(1-\omega)I_2 & \omega I_2 & 0 & \cdots & 0 & 0 \\
(1-\omega)\tilde{M} & \omega\tilde{M} + \tilde{N}_1 & \tilde{N}_2 & \cdots & \tilde{N}_{L-2} & \tilde{N}_{L-1} \\
0 & I_2 & 0 & \cdots & 0 & 0 \\
0 & 0 & I_2 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & I_2 & 0
\end{pmatrix}.
$$

Note that in $\tilde{Q}$ we have $\tilde{N}_i = \tilde{N}$ for all i and $\tilde{N}$ is zero except for element $\tilde{n}_{11}$. In the determinant eliminate the second column from each block ≥ 3

and also corresponding row. We get

$$\det[\tilde{Q} - \lambda I_{2L}] = (-\lambda)^{L-2} \det[\tilde{K}_{L+2}], \tag{67}$$

where

$$\tilde{K}_{L+2} = \begin{bmatrix} (1-\omega)I_2 - \lambda I_2 & \omega I_2 & 0 & 0 & \cdots & 0 & 0 \\ (1-\omega)\tilde{M} & \omega\tilde{M} + \tilde{N}_1 - \lambda I_2 & N1 & N1 & \cdots & N1 & N1 \\ 0 & (1,0) & -\lambda & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & -\lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & -\lambda & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & -\lambda \end{bmatrix}_{(L+2)x(L+2)}.$$

Consider first the case $L = 1$, so there are no lags. We can focus on the learning dynamics of π_t and R_t, i.e. the matrix

$$\begin{pmatrix} (1-\omega)I & \omega I \\ (1-\omega)\tilde{M} & \omega\tilde{M} \end{pmatrix}, \text{ where } \tilde{M} = \begin{pmatrix} \frac{-\pi^*}{(\beta-1)\beta\psi_p} & \frac{\beta\pi^*}{(\beta-1)\psi_p} \\ \frac{-1}{(\beta-1)\beta} & \frac{\beta}{\beta-1} \end{pmatrix}.$$

Assume $\psi_p > \beta^{-1}\pi^* = \bar{R}$. When $L = 1$ the system is four dimensional and two of the eigenvalues are those of $\tilde{M}$. Clearly $tr(\tilde{M} - I) < 0$ and $\det(\tilde{M} - I) > 0$. The other two eigenvalues are a repeated root equal to $1 - \omega < 1$ for all small ω. So E-stability holds in this case.

The characteristic polynomial of K_{L+2} has the following structure:[40]

$$\det[K_{L+2}] = \lambda(\lambda - 1 + \omega)P(n, \omega, \lambda)$$

where $n = L$ and

$$\begin{aligned} P(n,\omega,z) &= z^n + bz^{n-1} + cz^{n-2} + ... + cz + a, \text{ where} \\ b &= \frac{\omega}{1-\beta}b_1 \text{ with } b_1 = (1 - \frac{\pi^*}{\beta\psi_P}), \\ c &= \frac{\omega}{1-\beta} \text{ and } a = c - 1. \end{aligned} \tag{68}$$

Concerning the roots of the polynomial $\det[K_{L+2}] = 0$, there is one root equal to 0 and one equal to $1 - \omega$ which contribute to stability. The remaining n roots satisfy the equation

$$P(n, \omega, \lambda) = 0.$$

[40]The Mathematica routine is available on request.

With ω small local dynamics near the target steady state can be examined by looking at what happens to the roots of (68) as $\omega \to 0$. In the limit the polynomial equation becomes

$$\lambda^n = 1, \tag{69}$$

so the roots of the limit polynomial are given by the n roots of unity. If n is odd, 1 is the only real root of unity while if n is even, then both 1 and -1 are real roots. There are also complex roots. All roots are given by the list

$$\cos(2\pi k/n) + i * \sin(2\pi k/n), k = 0, 1, ..., n-1. \tag{70}$$

Note that $k = 0$ corresponds to real root 1 and in case n is even, the root -1 obtains for $k = n/2$. $P(n, \omega, \lambda)$ is now treated as a function of complex variables (ω, λ), where in fact ω is a small positive real. If λ is real, then resorting to complex values is not needed.

Consider first the simpler cases of a real root 1 or -1 (if n is even).

$$\frac{\partial P}{\partial \omega} = (1-\beta)^{-1}(b_1\lambda^{n-1} + \lambda^{n-2} + ... + \lambda + 1),$$

where $0 < b_1 < 1$. If λ is real

$$\frac{\partial P}{\partial \lambda} = n\lambda^{n-1} + \omega(1-\beta)^{-1}[(n-1)b_1\lambda^{n-2} + (n-2)c\lambda^{n-3} + ... + 2c\lambda + c].$$

At $\omega = 0$ and $\lambda = 1$ we have

$$\begin{aligned} \frac{\partial P}{\partial \omega} &= (1-\beta)^{-1}(b_1 + (n-1)) > 0 \\ \frac{\partial P}{\partial \lambda} &= n > 0. \end{aligned}$$

Then taking the differential of $P(n, \omega, \lambda) = 0$ and requiring

$$\frac{\partial P}{\partial \omega}d\omega + \frac{\partial P}{\partial \lambda}d\lambda = 0 \implies \frac{\partial \lambda}{\partial \omega} < 0.$$

So for small $\omega > 0$ the real root corresponding to limit 1 is inside the unit circle. At real root $\lambda = -1$ (now n is necessarily even) and $\omega \to 0$, so $\frac{\partial P}{\partial \omega} = (1-\beta)^{-1}(1-b_1) > 0$, and $\frac{\partial P}{\partial \lambda} = -n < 0$ and so $\frac{\partial \lambda}{\partial \omega} > 0$, i.e. for small $\omega > 0$ the real root approximate to -1 is inside the unit circle.

Consider now the case of complex roots and set $z = y + iy$ and so

$$P(n, \omega, z) = z^n + bz^{n-1} + cz^{n-2} + ... + cz + a, \tag{71}$$

and

$$\begin{aligned}\frac{\partial P}{\partial x} &= nz^{n-1} + b(n-1)z^{n-2} + (n-2)cz^{n-3} + ... + 2cz + c \\ \frac{\partial P}{\partial y} &= \left[nz^{n-1} + b(n-1)z^{n-2} + (n-2)cz^{n-3} + ... + 2cz + c\right] i.\end{aligned}$$

Thus

$$P' = \left(\frac{1}{2}\right)\left(\frac{\partial P}{\partial x} - i\frac{\partial P}{\partial y}\right) = nz^{n-1} + b(n-1)z^{n-2} + (n-2)cz^{n-3} + ... + 2cz + c$$

while the Cauchy-Riemann equation holds

$$\frac{\partial P}{\partial x} + i\frac{\partial P}{\partial y} = 0$$

as P is holomorphic.[41] Also compute

$$\frac{\partial P}{\partial \omega} = (1-\beta)^{-1}(b_1 z^{n-1} + z^{n-2} + ... + z + 1).$$

Introduce the notation $P' = 2(u+iv)$ and $\frac{\partial P}{\partial \omega} = (m+iq)$. The differential is

$$dP = \left(\frac{1}{2}\right) P' dz + \frac{\partial P}{\partial \omega} d\omega = (u+iv)(dx+idy) + (m+iq)d\omega.$$

Requiring $dP = 0$ is equivalent to $(u+iv)(dx+idy) + (m+iq)d\omega = 0$ which holds iff

$$\begin{aligned} udx - vdy + md\omega &= 0 \\ vdx + udy + qd\omega &= 0.\end{aligned}$$

In matrix form

$$\begin{pmatrix} u & -v \\ v & u \end{pmatrix}\begin{pmatrix} dx \\ dy \end{pmatrix} + \begin{pmatrix} m \\ q \end{pmatrix} d\omega = 0 \text{ so} \tag{72}$$

$$\begin{pmatrix} dx \\ dy \end{pmatrix} + (u^2+v^2)^{-1}\begin{pmatrix} u & v \\ -v & u \end{pmatrix}\begin{pmatrix} m \\ q \end{pmatrix} d\omega = 0.$$

[41] See Rudin (1970), Chapters 10 and 11 for the complex analysis used here.

We get

$$\begin{aligned}\frac{\partial x}{\partial \omega} &= -(u^2+v^2)^{-1}(um+vq) \\ \frac{\partial y}{\partial \omega} &= -(u^2+v^2)^{-1}(-vm+qu).\end{aligned}$$

A complex root $z = x+iy$ is inside the unit circle when $r^2 = x^2+y^2 < 1$. So if ω is varied from 0 to a small positive number correspondingly the complex root moves root of unity to inside unit circle if

$$\frac{\partial r^2}{\partial \omega} = x\frac{\partial x}{\partial \omega} + y\frac{\partial y}{\partial \omega} < 0.$$

Consider now the k'th root in (70) for a general n. Then $x = \cos(2\pi k/n)$, $y = \sin(2\pi k/n)$, $u = n\cos[2(n-1)\pi k/n]$ and $v = n\sin[2(n-1)\pi k/n]$ in the above formulae. Also $-(u^2+v^2)^{-1} = -n^{-2}$ and

$$\begin{aligned} m &= \left(b1\cos[2(n-1)\pi k/n] + \textstyle\sum_{s=2}^{n-1}\cos[2(n-s)\pi k/n] + 1\right)/(1-\beta) \\ q &= \left(b1\sin[2(n-1)\pi k/n] + \textstyle\sum_{s=2}^{n-1}\sin[2(n-s)\pi k/n]\right)/(1-\beta).\end{aligned}$$

In the case $L = n = 3$

$$P = (x+iy)^3 + \frac{\omega}{1-\beta}(b_1(x+iy)^2 + x+iy+1) - 1.$$

At $\omega = 0$ $P' = 3(x^2-y^2+2ixy)$ and $\frac{\partial P}{\partial \omega} = (1-\beta)^{-1}(b_1(x^2-y^2+2ixy)+x+iy+1)$. For (72) we get $u = 3(x^2+y^2)$, $v = 6xy$, $m = (1-\beta)^{-1}(b_1(x^2-y^2)+x+1)$ and $q = (1-\beta)^{-1}(2b_1xy+y)$. Using polar coordinates at complex roots of unity we have for $k = 1$ and 2 that $x = \cos(2\pi k/3)$, $y = \sin(2\pi k/3)$ which yields

$$\frac{\partial r^2}{\partial \omega} = -\frac{1(1-b_1)}{3(1-\beta)} < 0 \text{ when } k = 1 \text{ and } 2$$

implying that these roots contribute to stability. Using above arguments for real roots, stability also obtains at the real root that is approximate to 1.

Next consider the case $L = n = 4$. There are two real roots 1 and -1 and also two complex roots i and $-i$. It is easy to see that the real roots contribute to stability. At point $z = x+iy$ we have $P' = 4(x+iy)^3$ and

$\frac{\partial P}{\partial \omega} = (1-\beta)^{-1}(b_1(x+iy)^3 + (x+iy)^2 + x + iy + 1)$.So when $\omega = 0$ at the complex roots

$$\begin{aligned} P' &= -4i \text{ and } \frac{\partial P}{\partial \omega} = (1-\beta)^{-1}(1-b_1)i \text{ at root } z = i \text{ and} \\ P' &= 4i \text{ and } \frac{\partial P}{\partial \omega} = (1-\beta)^{-1}(b_1-1)i \text{ at root } z = -i. \end{aligned}$$

It follows that for $z = i$ $\frac{\partial x}{\partial \omega} = \frac{(1-b_1)}{4(1-\beta)} \equiv x_0 > 0$, at $z = -i$ $\frac{\partial x}{\partial \omega} = x_0 > 0$ and $\frac{\partial y}{\partial \omega} = 0$ for both roots. At root $z = i$ we have $\mathrm{mod}(z) = 1$, If ω becomes slightly positive, then the root changes to $\tilde{z} = x_0 dw + i$ with $\mathrm{mod}(\tilde{z}) = \sqrt{1 + (x_0 dw)^2} > 1$ implying instability. Analogous argument holds for $z = -i$.

Now consider the case $L = n = 5$. At point $z = x + iy$ we have $P' = 5(x+iy)^4$ and $\frac{\partial P}{\partial \omega} = (1-\beta)^{-1}(b_1(x+iy)^4 + (x+iy)^3 + (x+iy)^2 + x + iy + 1)$. So when $\omega = 0$ at the root $z = \cos(2\pi/5) + i * s \sin(2\pi/5)$, one gets

$$\frac{\partial r^2}{\partial \omega} = \frac{(\sqrt{5}-1)(1-b_1)}{10(1-\beta)} > 0$$

which implies that the root contributes to instability.

Now consider the case $L = n = 6$. At point $z = x + iy$ we have $P' = 6(x+iy)^5$ and $\frac{\partial P}{\partial \omega} = (1-\beta)^{-1}(b_1(x+iy)^5 + (x+iy)^4 + (x+iy)^3 + (x+iy)^2 + x + iy + 1)$. When $\omega = 0$ at the root $z = \cos(2\pi/6) + i * \sin(2\pi/6)$, one gets

$$\frac{\partial r^2}{\partial \omega} = \frac{(1-b_1)}{6(1-\beta)} > 0$$

which implies that the root contributes to instability.

Following our approach, one can show instability for higher values of $L > 6$.

Proof of the Remark in Section 2: Lag (3) and solve for $\hat{\pi}^e_{t-1}$ which gives

$$\hat{\pi}^e_{t-1} = M^{-1}\hat{\pi}_{t-1} + M^{-1}\sum_{i=2}^{L}\hat{\pi}_{t-i}, \text{ where } M = \frac{\pi^*}{\beta\psi}. \tag{73}$$

Assume $M < 1$. Then combine (4) and (73) into (3)

$$\begin{aligned}
\hat{\pi}_t &= M\left(\hat{\pi}^e_{t-1} + \omega(\hat{\pi}_{t-1} - \hat{\pi}^e_{t-1})\right) - \sum_{i=1}^{L-1} \hat{\pi}_{t-i} \\
&= M(1-\omega)\left(M^{-1}\hat{\pi}_{t-1} + M^{-1}\sum_{i=2}^{L} \hat{\pi}_{t-i}\right) + M\omega\hat{\pi}_{t-1} - \sum_{i=1}^{L-1} \hat{\pi}_{t-i} \\
&= \omega(M-1)\hat{\pi}_{t-1} - \omega\sum_{i=2}^{L-1} \hat{\pi}_{t-i} + (1-\omega)\hat{\pi}_{t-L}.
\end{aligned}$$

The characteristic polynomial is

$$P(n, \omega, z) = z^n + (1-M)\omega z^{n-1} + \omega z^{n-2} + ... + \omega z + (\omega - 1),$$

where $n = L$. In the limit $\omega \to 0$ the polynomial becomes

$$P = z^n - 1,$$

so there are $n-1$ roots of unity. Now at $\omega = 0$ we get $\frac{\partial P}{\partial z} = nz^{n-1}$ and $\frac{\partial P}{\partial \omega} = (1-M)z^{n-1} + z^{n-2} + ... + z + 1$.

Following the proof of Proposition 2 (i), we can show that $\partial z/\partial\omega = \frac{M-n}{n} < 0$ evaluated at $z = 1$, so the real root of 1 contributes to stability. Similarly, $\partial z/\partial\omega = \frac{M}{n} > 0$ evaluated at $z = -1$, so the real root of -1 contributes to stability (if n is even). A complex root $z = x + iy$ is inside the unit circle when $r^2 = x^2 + y^2 < 1$.

Consider now the case $L = n = 3$. There are two complex roots, and following the approach in the proof of Proposition 2(i), we can show

$$\frac{\partial r^2}{\partial \omega} = -\frac{M}{3} < 0$$

evaluated at either complex root, implying that these roots contribute to stability. From above, stability obtains at the real root that is approximate to 1.

Consider now the case $L = n = 4$. There are two complex roots, $z = i$ and $z = -i$, and following the approach in the proof of Proposition 2(i), we can show

$$\frac{\partial x}{\partial \omega} = \frac{M}{4} \equiv x_0 > 0 \qquad (74)$$

$$\frac{\partial y}{\partial \omega} = 0 \qquad (75)$$

evaluated at $z = i$ or $z = -i$. At root $z = i$ we have $\mathrm{mod}(z) = 1$, If ω becomes slightly positive, then the root changes to $\tilde{z} = x_0 d\omega + i$ with $\mathrm{mod}(\tilde{z}) = \sqrt{1 + (x_0 d\omega)^2} > 1$ implying instability. Analogous argument holds for $z = -i$.

We could consider also cases $L = 5, 6$ or higher. Figure A.2 gives a very long simulation for 50.000 periods in the example of Figure 1. Divergence is apparent.

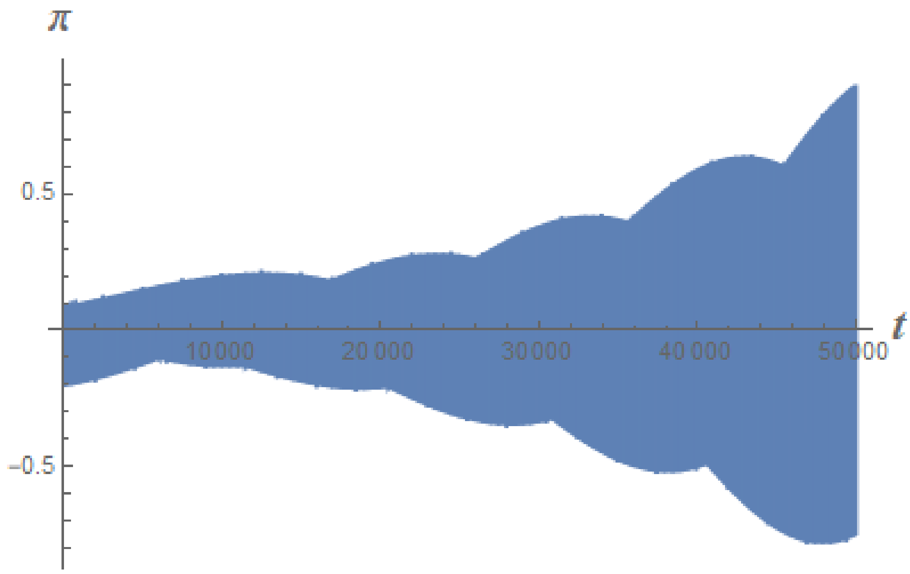


Figure A.2: Divergence in the Fisherian model for $L = 5$.

C.3 Proof of Proposition 3

The dynamic model is still given by the linearized system (25) and (26). Again in the limit $\gamma \to 0$ the first equation is independent from the rest of the system and output expectations y_t^e are convergent. Separating the equation for y_t^e, the state variables are $\tilde{x}_t = (\pi_t, R_t)^T$ and the linearized system is of the form (66) but the coefficient matrices $\tilde{M}$ and $\tilde{N}_i$ change to

$$\begin{aligned}\tilde{M} &= -(D\tilde{F}_x)^{-1}D\tilde{F}_{x^e} = \begin{pmatrix} \frac{\pi^*\left(\sum_{i=0}^{L-1}\mu^i\right)}{\psi_p\beta(1-\beta)} & -\frac{\beta\pi^*\left(\sum_{i=0}^{L-1}\mu^i\right)}{\psi_p(1-\beta)} \\ \frac{-1}{(\beta-1)\beta} & \frac{\beta}{\beta-1} \end{pmatrix}, \\ \tilde{N}_i &= -(D\tilde{F}_x)^{-1}D\tilde{F}_{x_i} = \begin{pmatrix} -\mu^i & 0 \\ 0 & 0 \end{pmatrix}, \; i = 1, ..., L-1.\end{aligned}$$

and the system is now

$$\tilde{Z}_t = \tilde{Q}_2\tilde{Z}_{t-1}, \tag{76}$$

where $\tilde{Z}_t$ is defined in the proof of Proposition 2, but $\tilde{Q}_2$ incorporates the new forms of $\tilde{M}$ and $\tilde{N}_i$ in $\tilde{Q}$. Consider the characteristic polynomial of $\tilde{Q}_2$ of (76)

$$\det[\tilde{Q}_2 - \lambda I_{2L}] = 0. \tag{77}$$

Given that the second columns of $\tilde{N}_i$ are zero vectors, the determinant in (77) has $L-2$ roots equal to zero. Then analyzing the remaining $(L+2)$ dimensional determinant, again it turns out that there is one more zero root and one root equal to $1-\omega$. Factoring out these, we are left with a polynomial of degree L. Introducing more familiar notation $n = L$, the polynomial is

$$P_2(n, \omega, \lambda) = \lambda^n + b(\omega)\lambda^{n-1} + a(\omega)[\mu\lambda^{n-2} ... + \mu^{n-2}\lambda] + (a(\omega)\mu^{n-1} - \mu^n), \tag{78}$$

where μ is the weight parameter in (28),

$$a(\omega) = \frac{\omega}{1-\beta} + \mu - 1, b(\omega) = a(\omega) - \frac{\omega}{1-\beta}b_1 \text{ with } b_1 = \frac{\pi^*\left(\sum_{i=0}^{n-1}\mu^i\right)}{\beta\psi_p}$$

and where $\pi^* < \beta\psi_P$ and $n \geq 2$ are assumed. We again consider how any root varies as ω varies from 0 to small values $d\omega > 0$ and require that in this variation the root is continuously a root of the characteristic polynomial. If $\omega \to 0$ we have $a \to \mu - 1$ and $b \to \mu - 1$, so the characteristic equation becomes

$$(1-\lambda)(\lambda^{n-1} + \mu\lambda^{n-2} + \mu^2\lambda^{n-3} + ... + \mu^{n-2}\lambda + \mu^{n-1}). \tag{79}$$

There is one root of unity. For the other roots one can apply a generalization of the classic Enerstrom-Kakeya theorem in Gardner and N.K. (2014), Theorem 3.6, stating that the other roots of the polynomial in (79) satisfy $|\lambda| < \mu < 1$.

Then consider the root of 1. Assume now a small perturbation $\omega > 0$. By continuity of eigenvalues the $n-1$ roots that are approximate to the roots of the latter polynomial in (79) remain inside the unit circle. To determine whether the unit root contributes to stability we compute the partial derivatives

$$\begin{aligned}
\frac{\partial P_2}{\partial \lambda} &= n\lambda^{n-1} + (n-1)b(\omega)\lambda^{n-2} + a(\omega)[\mu(n-2)\lambda^{n-3} ... + \mu^{n-2}], \\
\frac{\partial P_2}{\partial \omega} &= +b'(\omega)\lambda^{n-1} + a'(\omega)[\mu\lambda^{n-2} + ... + \mu^{n-2}\lambda] + a'(\omega)\mu^{n-1}.
\end{aligned}$$

At $\omega = 0$ and $\lambda = 1$ we have

$$\begin{aligned}
\frac{\partial P_2}{\partial \lambda} &= \frac{1-\mu^n}{1-\mu} > 0, \\
\frac{\partial P_2}{\partial \omega} &= \frac{1}{1-\beta}\left(1 - \frac{\pi^* \sum_{k=0}^{n-1} \mu^k}{\beta \psi_p} + \mu \frac{1-\mu^{n-1}}{1-\mu}\right) > 0,
\end{aligned}$$

since $a'(0) = (1-\beta)^{-1}$ and $b'(0) = (1-\beta)^{-1}(1-b_1)$. Then taking the differential of (78) and requiring

$$\frac{\partial P_2}{\partial \omega} d\omega + \frac{\partial P_2}{\partial \lambda} d\lambda = 0 \implies \frac{\partial \lambda}{\partial \omega} < 0.$$

So for small $\omega > 0$ the real root corresponding to limit 1 is inside the unit circle.

Next consider part (ii) of the proposition. In the linearization we get

$$DF_x = \begin{pmatrix} 1 & 0 & \frac{\beta(y^*-g)}{\pi^*\sigma} \\ -\kappa & 1 & 0 \\ -\frac{\psi_y}{y^*} & -\frac{\psi_p}{\pi^*\left(\sum_{i=0}^{L-1} \mu^i\right)} & 1 \end{pmatrix}$$

$$DF_{x^e} = \begin{pmatrix} -1 & \frac{-(g-y^*)}{\pi^*\sigma(\beta-1)} & \frac{\beta^2(g-y^*)}{\pi^*\sigma(\beta-1)} \\ \frac{\beta}{\beta-1}\kappa & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$DF_{x_{-i}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\frac{\mu^i\psi_p}{\pi^*\left(\sum_{i=0}^{L-1} \mu^i\right)} & 0 \end{pmatrix}, \; i = 1, ..., L-1,$$

where

$$\kappa = \frac{\nu\left(\frac{(\nu-1)\sigma y^*(y^*-\bar{g})^{-\sigma-1}}{\nu} - \frac{(\nu-1)(y^*-\bar{g})^{-\sigma}}{\nu} + \frac{(\epsilon+1)y*^{\frac{\epsilon+1}{\alpha}-1}}{\alpha^2}\right)}{\gamma(2\pi^*-1)} \geq 0.$$

It follows that

$$M = -(DF_x)^{-1}DF_{x^e} =$$

$$\begin{pmatrix} \frac{y^*\left(\beta^2\kappa\psi_p(y^*-\bar{g})/\left(\sum_{i=0}^{L-1}\mu^i\right)+(\beta-1)(\pi^*)^2\sigma\right)}{a} & \frac{\pi^*y^*(\bar{g}-y^*)}{a} & \frac{\beta^2\pi^*y^*(y^*-\bar{g})}{a} \\ \frac{\kappa(-\pi^*)\left(\beta^2\psi_y(y^*-\bar{g})+\pi^*\sigma y^*\right)}{a} & \frac{\kappa\pi^*y^*(\bar{g}-y^*)}{a} & \frac{\beta^2\kappa\pi^*y^*(y^*-\bar{g})}{a} \\ \frac{\pi^*\sigma((\beta-1)\pi^*\psi_y-\kappa\psi_p y^*/\left(\sum_{i=0}^{L-1}\mu^i\right))}{a} & \frac{(\bar{g}-y^*)(\pi^*\psi_y+\kappa\psi_p y^*)}{a} & \frac{\beta^2(y^*-\bar{g})(\pi^*\psi_y+\kappa\psi_p y^*)}{a} \end{pmatrix},$$

$$N_i = -(DF_x)^{-1}DF_{x_i} =$$

$$\begin{pmatrix} 0 & \frac{\mu^i\beta\psi_p y^*(\bar{g}-y^*)(\beta-1)}{\left(\sum_{i=0}^{L-1}\mu^i\right)a} & 0 \\ 0 & \frac{\mu^i\beta\kappa\psi_p y^*(\bar{g}-y^*)(\beta-1)}{\left(\sum_{i=0}^{L-1}\mu^i\right)a} & 0 \\ 0 & \frac{\mu^i\pi^*\sigma\psi_p y^*(\beta-1)}{\left(\sum_{i=0}^{L-1}\mu^i\right)a} & 0 \end{pmatrix}, \; i = 1, ..., L-1.$$

where

$$a = (\beta-1)\left(\pi^*(\beta\psi_y(y^*-\bar{g}) + \pi^*\sigma y^*) + \beta\kappa\psi_p y^*(y^*-\bar{g})/\left(\sum_{i=0}^{L-1}\mu^i\right)\right) < 0.$$

The system is now like (65)

$$Z_t = Q_2 Z_{t-1}, \tag{80}$$

where Z_t is as before in Proposition 1, but Q_2 incorporates the new forms of M and N_i. Introduce the notation $x_t = (y_t, \pi_t, R_t)$ etc. Modifying the system yields

$$Z_t = (\; x_t^e \quad x_t \quad x_{t-1} \quad x_{t-2} \quad \cdots \quad x_{t-(L-2)} \;)^T$$

$$Q_2 = \begin{pmatrix} (1-\omega)I_3 & \omega I_3 & 0 & \cdots & 0 & 0 \\ (1-\omega)M & \omega M + N_1 & N_2 & \cdots & N_{L-2} & N_{L-1} \\ 0 & I_3 & 0 & \cdots & 0 & 0 \\ 0 & 0 & I_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I_3 & 0 \end{pmatrix}.$$

For stability, the roots of $P(\lambda) = Det[Q_2 - \lambda I_{3L}]$ must be inside the unit circle. One can show that

$$P(\lambda) = \lambda^{2L-2}(1+\omega-\lambda)\tilde{P}(\lambda)$$

Thus, the roots of $P(\lambda)$ are inside the unit circle if and only if the roots of $\tilde{P}(\lambda)$ are inside the unit circle. In the limit $\omega \to 0$, we have

$$\tilde{P}(\lambda) = (1-\lambda)^2(\lambda^{L-1} + h\mu \sum_{k=0}^{L-2} \mu^{L-2-k}\lambda^k)$$

where

$$h = \frac{\beta\kappa\psi_p y^*(y^* - \bar{g})}{\beta\kappa\psi_p y^*(y^* - \bar{g}) + \left(\beta\pi^*\psi_y(y^* - \bar{g}) + (\pi^*)^2\sigma y^*\right)\left(\sum_{i=0}^{L-1}\mu^i\right)} \in (0,1)$$

The polynomial has two unit roots. For the other roots one can apply a generalization of the classic Enerstrom-Kakeya theorem in Gardner and N.K. (2014), Theorem 3.6, stating that the roots of the second polynomial in $\tilde{P}(\lambda)$ satisfy $|\lambda| < \mu < 1$.

Therefore, the roots of $P(\lambda)$ are inside the unit circle if $\partial\lambda/\partial\omega < 0$ evaluated at $\omega = 0$ and $\lambda = 1$. To evaluate the derivative, we consider the Taylor series expansion of $\tilde{P}(\lambda)$ up to second order at point (λ_0, ω_0). Let $(d\lambda, d\omega) = (\lambda, \omega) - (\lambda_0, \omega_0)$. Then

$$\begin{aligned}\tilde{P}(\lambda,\omega) &= \tilde{P}(\lambda_0,\omega_0) + \tilde{P}_\lambda(\lambda_0,\omega_0)d\lambda + \tilde{P}_\omega(\lambda_0,\omega_0)d\omega + \\ &\quad \tilde{P}_{\lambda\lambda}(\lambda_0,\omega_0)\frac{d\lambda^2}{2} + \tilde{P}_{\lambda\omega}(\lambda_0,\omega_0)d\lambda d\omega + \tilde{P}_{\omega\omega}(\lambda_0,\omega_0)\frac{d\omega^2}{2} + Q,\end{aligned}$$

where subscripts denote partial derivatives and Q is a remainder.

Now

$$\begin{aligned}\tilde{P}_\omega(\lambda_0,\omega_0) &= 0 \\ \tilde{P}_\lambda(\lambda_0,\omega_0) &= 0\end{aligned}$$

so we get the approximation

$$\tilde{P}(\lambda,\omega) = \tilde{P}(\lambda_0,\omega_0) + \tilde{P}_{\lambda\lambda}(\lambda_0,\omega_0)\frac{d\lambda^2}{2} + \tilde{P}_{\lambda\omega}(\lambda_0,\omega_0)d\lambda d\omega + \tilde{P}_{\omega\omega}(\lambda_0,\omega_0)\frac{d\omega^2}{2}.$$

Now impose

$$\tilde{P}(\lambda,\omega)-\tilde{P}(\lambda_0,\omega_0)=0$$

to compute the derivative of the implicit function. So we have

$$\tilde{P}_{\lambda\omega}(\lambda_0,\omega_0)d\lambda d\omega+\tilde{P}_{\omega\omega}(\lambda_0,\omega_0)\frac{d\omega^2}{2}+\tilde{P}_{\lambda\lambda}(\lambda_0,\omega_0)\frac{d\lambda^2}{2}=0$$

or

$$\frac{d\lambda}{d\omega}=\frac{-1}{2}\left(\frac{\tilde{P}_{\omega\omega}(\lambda_0,\omega_0)}{\tilde{P}_{\lambda\omega}(\lambda_0,\omega_0)}+\frac{\tilde{P}_{\lambda\lambda}(\lambda_0,\omega_0)}{\tilde{P}_{\lambda\omega}(\lambda_0,\omega_0)}\left(\frac{d\lambda}{d\omega}\right)^2\right)$$

Evaluating the partial derivatives at $(\lambda_0,\omega_0)=(1,0)$ we have

$$\begin{aligned}
\tilde{P}_{\omega\omega}(1,0) &= (-1)^L\frac{2(y^*-\bar{g})((1-\beta)\beta\pi^*\psi_y+\kappa y^*(\beta\psi_p-\pi^*))}{(\beta-1)^2\left(\beta\pi^*\psi_y(y^*-\bar{g})+\beta\kappa\psi_p y^*(y^*-\bar{g})/\left(\sum_{k=0}^{L-1}\mu^i\right)+(\pi^*)^2\sigma y^*\right)}\\
\tilde{P}_{\lambda\lambda}(1,0) &= (-1)^L\frac{2\left(\beta\pi^*\psi_y(y^*-\bar{g})+\beta\kappa\psi_p y^*(y^*-\bar{g})+(\pi^*)^2\sigma y^*\right)}{\beta\pi^*\psi_y(y^*-\bar{g})+\beta\kappa\psi_p y^*(y^*-\bar{g})/\left(\sum_{k=0}^{L-1}\mu^i\right)+(\pi^*)^2\sigma y^*}\\
\tilde{P}_{\lambda\omega}(1,0) &= (-1)^L\frac{\kappa y^*(y^*-\bar{g})(\pi^*-2\beta\psi_p)+(2-\beta)\beta\pi^*\psi_y(\bar{g}-y^*)-(1-\beta)(\pi^*)^2\sigma y^*}{(\beta-1)\left(\beta\pi^*\psi_y(y^*-\bar{g})+\beta\kappa\psi_p y^*(y^*-\bar{g})/\left(\sum_{k=0}^{L-1}\mu^i\right)+(\pi^*)^2\sigma y^*\right)}
\end{aligned}$$

One can show that $\tilde{P}_{\omega\omega}(1,0)>0$, $\tilde{P}_{\lambda\lambda}(1,0)>0$, $\tilde{P}_{\lambda\omega}(1,0)>0$ if L is even and $\tilde{P}_{\omega\omega}(1,0)<0$, $\tilde{P}_{\lambda\lambda}(1,0)<0$, $\tilde{P}_{\lambda\omega}(1,0)<0$ if L is odd. Therefore, $\partial\lambda/\partial\omega<0$ and we have stability for $\kappa\geq 0$ and small ω.

C.4 Proof of Proposition 4

In the linearization (32) we get

$$DF_x=\begin{pmatrix} 1 & 0 & \frac{\beta(y^*-g)}{\pi^*\sigma}\\ -\kappa & 1 & 0\\ -\frac{\psi_y}{y^*} & -\frac{w_c\psi_p}{\pi^*} & 1\end{pmatrix}$$

$$DF_{x^e}=\begin{pmatrix} -1 & \frac{-(g-y^*)}{\pi^*\sigma(\beta-1)} & \frac{\beta^2(g-y^*)}{\pi^*\sigma(\beta-1)}\\ \frac{\beta\kappa}{\beta-1} & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

$$DF_{cb} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -(1-w_c)\psi_p/\pi^* & 0 \end{pmatrix}$$

where

$$\kappa = \frac{\nu\left(\frac{(\nu-1)\sigma y^*(y^*-\bar{g})^{-\sigma-1}}{\nu} - \frac{(\nu-1)(y^*-\bar{g})^{-\sigma}}{\nu} + \frac{(\epsilon+1)y*^{\frac{\epsilon+1}{\alpha}-1}}{\alpha^2}\right)}{\gamma(2\pi^*-1)} \geq 0$$

if $\sigma > (y^* - \bar{g})/y^*$. It follows that

$$\begin{aligned} M &= -(DF_x)^{-1}DF_{x^e} = \\ &\begin{pmatrix} \frac{y^*\left(w_c\beta^2\kappa\psi_p(y^*-\bar{g})+(\beta-1)\pi*^2\sigma\right)}{a} & \frac{\pi^*y^*(\bar{g}-y^*)}{a} & \frac{\beta^2\pi^*y^*(y^*-\bar{g})}{a} \\ \frac{\kappa(-\pi^*)\left(\beta^2\psi_y(y^*-\bar{g})+\pi^*\sigma y^*\right)}{a} & \frac{\kappa\pi^*y^*(\bar{g}-y^*)}{a} & \frac{\beta^2\kappa\pi^*y^*(y^*-\bar{g})}{a} \\ \frac{\pi^*\sigma((\beta-1)\pi^*\psi_y-w_c\kappa\psi_py^*)}{a} & \frac{(\bar{g}-y^*)(\pi^*\psi_y+w_c\kappa\psi_py^*)}{a} & \frac{\beta^2(y^*-\bar{g})(\pi^*\psi_y+w_c\kappa\psi_py^*)}{a} \end{pmatrix}, \\ N &= -(DF_x)^{-1}DF_{cb} = \\ &\begin{pmatrix} 0 & \frac{(w_c-1)\beta\psi_py^*(\bar{g}-y^*)}{\left(\beta\pi^*\psi_y(y^*-\bar{g})+w_c\beta\kappa\psi_py^*(y^*-\bar{g})+(\pi^*)^2\sigma y^*\right)} & 0 \\ 0 & \frac{(w_c-1)\beta\kappa\psi_py^*(\bar{g}-y^*)}{\left(\beta\pi^*\psi_y(y^*-\bar{g})+w_c\beta\kappa\psi_py^*(y^*-\bar{g})+(\pi^*)^2\sigma y^*\right)} & 0 \\ 0 & \frac{(w_c-1)\pi^*\sigma\psi_py^*}{\left(\beta\pi^*\psi_y(y^*-\bar{g})+w_c\beta\kappa\psi_py^*(y^*-\bar{g})+(\pi^*)^2\sigma y^*\right)} & 0 \end{pmatrix} \\ N_{cb} &= \\ &\begin{pmatrix} \frac{(w_c-1)\beta\psi_py^*(\bar{g}-y^*)}{\left(\beta\pi^*\psi_y(y^*-\bar{g})+w_c\beta\kappa\psi_py^*(y^*-\bar{g})+(\pi^*)^2\sigma y^*\right)} \\ \frac{(w_c-1)\beta\kappa\psi_py^*(\bar{g}-y^*)}{\left(\beta\pi^*\psi_y(y^*-\bar{g})+w_c\beta\kappa\psi_py^*(y^*-\bar{g})+(\pi^*)^2\sigma y^*\right)} \\ \frac{(w_c-1)\pi^*\sigma\psi_py^*}{\left(\beta\pi^*\psi_y(y^*-\bar{g})+w_c\beta\kappa\psi_py^*(y^*-\bar{g})+(\pi^*)^2\sigma y^*\right)} \end{pmatrix} \end{aligned}$$

where $a = (\beta - 1)\left(\beta\pi^*\psi_y(y^* - \bar{g}) + w_c\beta\kappa\psi_p y^*(y^* - \bar{g}) + (\pi^*)^2\sigma y^*\right) < 0$.

Introduce the notation $x_t = (y_t, \pi_t, R_t)$ etc. Modifying the system (25), (32), and the linearization of (30) yields yields

$$Z_t = QZ_{t-1}, \text{ where} \tag{81}$$

$$Z_t = \begin{pmatrix} x_t & x_t^e & \pi_t^{cb} \end{pmatrix}^T$$

$$Q = \begin{pmatrix} \omega M + w_c N & (1-\omega)M & (1-w_c)N_{cb} \\ \omega I_3 & (1-\omega)I_3 & 0_{3\times 1} \\ 0 \;\; w_c \;\; 0 & \cdots & 1-w_c \end{pmatrix}.$$

For stability, the roots of $P(\lambda) = Det[Q - \lambda I_7]$ must be inside the unit circle. One can show that

$$P(\lambda) = \lambda^3(1 + \omega - \lambda)\tilde{P}(\lambda).$$

Thus, the roots of $P(\lambda)$ are inside the unit circle if and only if the roots of $\tilde{P}(\lambda)$ are inside the unit circle. In the limit $\omega \to 0$, we have

$$\tilde{P}(\lambda) = (1 - \lambda)^2(\lambda - \mu)$$

where $\mu = \frac{(1-w_c)\left(\beta\pi^*\psi_y(y^*-\bar{g})+(\pi^*)^2\sigma y^*\right)}{(y^*-\bar{g})(\beta\pi^*\psi_y+\beta\kappa w_c\psi_p y^*)+(\pi^*)^2\sigma y^*} < \frac{\left(\beta\pi^*\psi_y(y^*-\bar{g})+(\pi^*)^2\sigma y^*\right)}{(y^*-\bar{g})(\beta\pi^*\psi_y)+(\pi^*)^2\sigma y^*} = 1$. Therefore, the roots of $P(\lambda)$ are inside the unit circle if $\partial\lambda/\partial\omega < 0$ evaluated at $\omega = 0$ and $\lambda = 1$. To evaluate the derivative, we consider the Taylor series expansion of $\tilde{P}(\lambda)$ up to second order at point (λ_0, ω_0). Let $(d\lambda, d\omega) = (\lambda, \omega) - (\lambda_0, \omega_0)$. Then

$$\begin{aligned}\tilde{P}(\lambda, \omega) &= \tilde{P}(\lambda_0, \omega_0) + \tilde{P}_\lambda(\lambda_0, \omega_0)d\lambda + \tilde{P}_\omega(\lambda_0, \omega_0)d\omega + \\ &\quad \tilde{P}_{\lambda\lambda}(\lambda_0, \omega_0)\frac{d\lambda^2}{2} + \tilde{P}_{\lambda\omega}(\lambda_0, \omega_0)d\lambda d\omega + \tilde{P}_{\omega\omega}(\lambda_0, \omega_0)\frac{d\omega^2}{2} + Q,\end{aligned}$$

where subscripts denote partial derivatives and Q is a remainder.

Evaluating the partial derivatives at $(\lambda_0, \omega_0) = (1, 0)$ we have

$$\begin{aligned}\tilde{P}_\omega(1, 0) &= 0, \\ \tilde{P}_\lambda(1, 0) &= 0,\end{aligned}$$

and imposing

$$\tilde{P}(\lambda, \omega) - \tilde{P}(\lambda_0, \omega_0) = 0,$$

we get the approximation

$$\tilde{P}_{\lambda\omega}(1, 0)d\lambda dw + \tilde{P}_{\omega\omega}(1, 0)\frac{d\omega^2}{2} + \tilde{P}_{\lambda\lambda}(1, 0)\frac{d\lambda^2}{2} = 0$$

or

$$\frac{d\lambda}{d\omega} = \frac{-1}{2}\left(\frac{\tilde{P}_{\omega\omega}(1, 0)}{\tilde{P}_{\lambda\omega}(1, 0)} + \frac{\tilde{P}_{\lambda\lambda}(1, 0)}{\tilde{P}_{\lambda\omega}(1, 0)}\left(\frac{d\lambda}{d\omega}\right)^2\right)$$

Further, we have

$$
\begin{aligned}
\tilde{P}_{\omega\omega}(1,0) &= \frac{2w_c(y^*-\bar{g})((\beta-1)\beta\pi^*\psi_y+\kappa\pi^*y^*-\beta\kappa\psi_p y^*)}{(1-\beta)^{-1}\left(\beta\pi^*\psi_y(y^*-\bar{g})+\beta\kappa w_c\psi_p y^*(y^*-\bar{g})+(\pi^*)^2\sigma y^*\right)} \\
\tilde{P}_{\lambda\lambda}(1,0) &= -\frac{2w_c\left(\beta\pi^*\psi_y(y^*-\bar{g})+\beta\kappa\psi_p y^*(y^*-\bar{g})+(\pi^*)^2\sigma y^*\right)}{\left(\beta\pi^*\psi_y(y^*-\bar{g})+\beta\kappa w_c\psi_p y^*(y^*-\bar{g})+(\pi^*)^2\sigma y^*\right)} \\
\tilde{P}_{\lambda\omega}(1,0) &= \frac{w_c y^*\left((\beta-1)(\pi^*)^2\sigma+(\beta-2)\beta\pi^*\psi_y+\kappa\pi^*y^*-2\beta\kappa\psi_p y^*\right)}{(1-\beta)\left(\beta\pi^*\psi_y(y^*-\bar{g})+\beta\kappa w_c\psi_p y^*(y^*-\bar{g})+(\pi^*)^2\sigma y^*\right)} \\
&- \frac{\bar{g}w_c((\beta-2)\beta\pi^*\psi_y+\kappa\pi^*y^*-2\beta\kappa\psi_p y^*)}{(1-\beta)\left(\beta\pi^*\psi_y(y^*-\bar{g})+\beta\kappa w_c\psi_p y^*(y^*-\bar{g})+(\pi^*)^2\sigma y^*\right)}
\end{aligned}
$$

One can show that $\tilde{P}_{\omega\omega}(1,0)<0$, $\tilde{P}_{\lambda\lambda}(1,0)<0$, $\tilde{P}_{\lambda\omega}(1,0)<0$ if $\beta\psi_p>\pi^*$. Therefore, $\partial\lambda/\partial\omega<0$ and we have stability for small w and $\kappa>0$.

In part (ii) with $\gamma=0$ the dynamics of output expectations do not depend on the rest of the system and can be shown to be locally convergent. The linearization (32) becomes

$$
\begin{aligned}
\tilde{M} &\equiv -(D\tilde{F}_x)^{-1}D\tilde{F}_{x^e} = \begin{pmatrix} \frac{\pi^*}{w_c\psi_p\beta(1-\beta)} & -\frac{\beta\pi^*}{w_c\psi_p(1-\beta)} \\ \frac{1}{\beta(1-\beta)} & -\frac{\beta}{(1-\beta)} \end{pmatrix} \text{ and} \\
\tilde{N} &\equiv -(D\tilde{F}_x)^{-1}D\tilde{F}_{x_i^{\cdot}} = \begin{pmatrix} \frac{w_c-1}{w_c} & 0 \\ 0 & 0 \end{pmatrix} \text{ and} \\
\tilde{N}_{cb} &\equiv -(D\tilde{F}_x)^{-1}D\tilde{F}_{x_i^{\cdot}} = \begin{pmatrix} \frac{w_c-1}{w_c} \\ 0 \end{pmatrix},\ i=1,...,L-1.
\end{aligned}
$$

Introduce the notation $\tilde{x}_t=(\pi_t,R_t)$ etc. Modifying the system (25), (32) and the linearization of (30) yields

$$
\tilde{Z}_t=\tilde{Q}\tilde{Z}_{t-1},\text{ where} \tag{82}
$$

$$
\tilde{Z}_t = \begin{pmatrix} x_t & x_t^e & \pi_t^{cb} \end{pmatrix}^T
$$

$$
\tilde{Q} = \begin{pmatrix} \omega\tilde{M}+w_c\tilde{N} & (1-\omega)\tilde{M} & (1-w_c)\tilde{N}_{cb} \\ \omega I_2 & (1-\omega)I_2 & 0_{2\times1} \\ w_c\ \ 0 & \cdots & 1-w_c \end{pmatrix}.
$$

For stability, the roots of $P(\lambda)=Det[\tilde{Q}-\lambda I_5]$ must be inside the unit circle. One can show that

$$
P(\lambda)=\lambda^2(1+\omega-\lambda)\tilde{P}(\lambda),
$$

where

$$\tilde{P}(\lambda) = \lambda^2 + \frac{\beta w_c \psi_p(\beta + \omega - 1) - \pi^* \omega}{(1-\beta)\beta w_c \psi_p}\lambda + \frac{\pi^* \omega(1 - w_c)}{(1-\beta)\beta w_c \psi_p}.$$

Thus, the roots of $P(\lambda)$ are inside the unit circle if and only if the roots of $\tilde{P}(\lambda)$ are inside the unit circle.

Let $a_0 = \frac{\pi^*\omega(1-w_c)}{(1-\beta)\beta w_c \psi_p}$ and $a_1 = \frac{\beta w_c \psi_p(\beta+\omega-1)-\pi^*\omega}{(1-\beta)\beta w_c \psi_p}$. The roots of $\tilde{P}(\lambda)$ are inside the unit circle if and only if the Schur-Cohn condition, $|a_1| < 1 + a_0 < 2$, is satisfied. The Schur-Cohn condition is satisfied if $\psi_p > \max[\frac{\pi^*(\omega/w_c)(1-w_c)}{(1-\beta)\beta}, \bar{R}]$ and $\omega < \frac{(1-\beta)\beta w_c \psi_p}{\beta w_c \psi_p - \pi^*}$ if $\psi_p > \pi^*/(\beta w_c)$ or $\omega > 0$ otherwise.

References

ANDRADE, B., J. GALI, H. LE BIHAN, AND J. MATHERON (2021): "Should the ECB Adjust its Strategy in the Face of a Lower r*?," mimeo.

BENHABIB, J., G. W. EVANS, AND S. HONKAPOHJA (2014): "Liquidity Traps and Expectation Dynamics: Fiscal Stimulus or Fiscal Austerity?," *Journal of Economic Dynamics and Control*, 45, 220–238.

BERNANKE, B. (2017): "Monetary Policy in a New Era," Peterson institute conference on rethinking macroeconomic policy, october 2017.

BHATTARAI, S., AND C. NEELY (2016): "A Survey of the Empirical Literature on U.S. Unconventional Monetary," Working paper, Federal Reserve Bank of St. Louis Working Paper 2016-021A.

BRANCH, W. (2006): "Restricted Perceptions Equilibria and Learning in Macroeconomics," in Colander (2006), pp. 135–160.

BRANCH, W. A., AND G. W. EVANS (2006): "A Simple Recursive Forecasting Model," *Economic Letters*, 91, 158–166.

BUDIANTO, F., D. NAKATO, AND S. SCHMIDT (2020): "Average Inflation Targeting and the Interest Rate Lower Bound," Working paper no.852, BIS.

BULLARD, J. (2018): "Speech: A Primer on Price Level Targeting in the U.S.," Slides.

BULLARD, J., AND K. MITRA (2002): "Learning About Monetary Policy Rules," *Journal of Monetary Economics*, 49, 1105–1129.

CALVO, G. A. (1983): "Staggered Pricing in a Utility-Maximizing Framework," *Journal of Monetary Economics*, 12, 383–398.

CARNEY, M. (2012): "Guidance," Speech, www.bankofcanada.ca.

COBHAM, D., Ø. EITRHEIM, S. GERLACH, AND J. F. QVIGSTAD (eds.) (2010): *Twenty Years of Inflation Targeting: Lessons Learned and Future Prospects*. Cambridge University Press, Cambridge.

Colander, D. (2006): *Post Walrasian Macroeconomics*. Cambridge, Cambridge, U.K.

Dell'Ariccia, G., P. Rabanal, and D. Sandri (2018): "Unconventional Monetary Policies in the Euro Area, Japan, and the United Kingdom," *Journal of Economic Perspectives*, 32.

Eusepi, S., M. Giannoni, and B. Preston (2018): "Some Implications of Learning for Price Stability," *European Economic Review*, 106, 1–20.

Eusepi, S., and B. Preston (2018): "The Science of Monetary Policy: An Imperfect Knowledge Perspective," *Journal of Economic Literature*, 56, 3–59.

Evans, C. L. (2012): "Monetary Policy in a Low-Inflation Environment: Developing a State-Contingent Price-Level Target," *Journal of Money, Credit and Banking*, 44, 147–155.

Evans, G. W., E. Guse, and S. Honkapohja (2008): "Liquidity Traps, Learning and Stagnation," *European Economic Review*, 52, 1438–1463.

Evans, G. W., and S. Honkapohja (2001): *Learning and Expectations in Macroeconomics*. Princeton University Press, Princeton, New Jersey.

——— (2003): "Expectations and the Stability Problem for Optimal Monetary Policies," *Review of Economic Studies*, 70, 807–824.

——— (2006): "Monetary Policy, Expectations and Commitment," *Scandinavian Journal of Economics*, 108, 15–38.

——— (2009a): "Learning and Macroeconomics," *Annual Review of Economics*, 1, 421–451.

——— (2009b): "Robust Learning Stability with Operational Monetary Policy Rules," in Schmidt-Hebbel and Walsh (2009), chap. 5, pp. 145–170.

——— (2010): "Expectations, Deflation Traps and Macroeconomic Policy," in Cobham, Eitrheim, Gerlach, and Qvigstad (2010), chap. 12, pp. 232–260.

——— (2013): "Learning as a Rational Foundation for Macroeconomics and Finance," in Frydman and Phelps (2013), chap. 2, pp. 68–111.

Evans, G. W., S. Honkapohja, and K. Mitra (2012): "Does Ricardian Equivalence Hold When Expectations are not Rational?," *Journal of Money, Credit and Banking*, 44, 1259–1283.

——— (2020): "Expectations, Stagnation and Fiscal Policy: a Nonlinear Analysis," Discussion paper 15171, CEPR.

Frydman, R., and E. e. Phelps (eds.) (2013): *Rethinking Expectations: The Way Forward for Macroeconomics*. Princeton University Press, Princeton, New Jersey.

Gali, J. (2008): *Monetary Policy, Inflation, and the Business Cycle*. Princeton University Press, Princeton New Jersey.

Gardner, R., and G. N.K. (2014): "Enestrom-Kakeya Theorem and Some of Its Generalizations," in Joshi, Dorff, and Lahiri (2014), pp. 1–33.

Hebden, J., P. Herbst, J. Tang, G. Topa, and F. Winker (2020): "How Robust Are Makeup Strategies to Key Alternative Assumptions?," Finance and economics discussion series 2020-069. washington: Board of governors of the federal reserve system, https://doi.org/10.17016/feds.2020.069.

Honkapohja, S., and K. Mitra (2020): "Price Level Targeting with Evolving Credibility," *Journal of Monetary Economics*, 116, 88–103, https://doi.org/10.1016/j.jmoneco.2019.09.009.

Honkapohja, S., K. Mitra, and G. W. Evans (2013): "Notes on Agents' Behavioral Rules Under Adaptive Learning and Studies of Monetary Policy," in Sargent and Vilmunen (2013), chap. 5.

Joshi, S., M. Dorff, and I. Lahiri (eds.) (2014): *Current Topics in Pure and Computational Complex Analysis*. Springer-India.

Jury, E. (1961): "A Simplified Stability Criterion for Linear Discrete Systems," *Proceedings of the IRE*, 49, 1493–1500.

Kuttner, K. (2018): "Outside the Box: Unconventional Monetary Policy in the Great Recession and Beyond," *Journal of Economic Perspectives*, 32.

Leeper, E. M. (1991): "Equilibria under 'Active' and 'Passive' Monetary and Fiscal Policies," *Journal of Monetary Economics*, 27, 129–147.

Mertens, T. M., and J. C. Williams (2019): "Tying Down the Anchor: Monetary POolicy Rules and the Lower Bound on Interest Rates," Staff report no.887, Federal Reserve Bank of New York.

Milani, F. (2007): "Expectations, Learning and Macroeconomic Persistence," *Journal of Monetary Economics*, 54, 2065–2082.

Moessner, R., D.-J. Jansen, and J. de Haan (2017): "Communication about future policy rates in theory and practice: a survey," *Journal of Economic Surveys*, 31.

Nessen, M., and D. Vestin (2005): "Average Inflation Targeting," *Journal of Money, Credit and Banking*, 37, 837–863.

Orphanides, A., and J. C. Williams (2005): "Inflation Scares and Forecast-Based Monetary Policy," *Review of Economic Dynamics*, 8, 498–527.

Powell, J. H. (2020): "New Economic Challenges and the Fed Monetary Policy Review," Remarks.

Reifschneider, D., and J. C. Williams (2000): "Three Lessons for Monetary Policy in a Low-Inflation Era," *Journal of Money, Credit and Banking*, 32, 936–966.

Rotemberg, J. J. (1982): "Sticky Prices in the United States," *Journal of Political Economy*, 90, 1187–1211.

Rudin, W. (1970): *Real and Complex Analysis*. McGraw-Hill, New York.

Sargent, T. J. (1999): *The Conquest of American Inflation*. Princeton University Press, Princeton NJ.

——— (2008): "Evolution and Intelligent Design," *American Economic Review*, 98, 5–37.

Sargent, T. J., and J. Vilmunen (eds.) (2013): *Macroeconomics at the Service of Public Policy*. Oxford University Press.

Schmidt-Hebbel, K., and C. E. Walsh (eds.) (2009): *Monetary Policy under Uncertainty and Learning*. Central Bank of Chile, Santiago.

Svensson, L. E. (2020): "Monetary Policy Strategies for the Federal Reserve," *International Journal of Central Banking*, 16, 133–193.

Williams, J. C. (2017): "Preparing for the Next Storm: Reassessing Frameworks and Strategies in a Low R-star World," *FRB Economic Letter 2017-13*, pp. 1–8.