Common Value Auctions with Costly Entry

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Abstract

We analyze an affiliated common values auction with costly participation with an unknown number of competing bidders. We call such auctions informal auctions. We contrast the symmetric equilibria of informal first-price auctions with the well-understood symmetric equilibria of formal auctions where the number of entrants is common knowledge at the bidding stage. With endogenous entry, the informal first-price auction often yields a higher expected payoff than any of the standard formal auctions. JEL Classification: D44

1 Introduction

Entering an auction often entails significant costs. In addition to the opportunity cost of time and effort spent on being physically present at an auction site, acquiring information about one's own valuation for the good is often costly. On top of this, the preparation of bids may be costly as a result of concerns for due diligence. If entry costs are sunk at a stage where the eventual number of participants in the auction is unknown, it is natural to consider the performance of various auction formats with a random number of bidders. We call such auctions informal auctions.

In this paper, we show that accounting for this uncertainty leads to new types of bidding equilibria and to new revenue results in single object auctions with affiliated common values. In particular, we demonstrate the superior performance of first-price auctions in common value auctions with an undisclosed number of bidders when entry costs are significant.

The previous literature on auctions with costly participation has focused on sequential decisions by the bidders. Entry decisions are taken in the first stage. In the second stage, all entering bidders see the realized number of participants and bid optimally in the ensuing auction. In this paper,

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we relax the assumption that the set of bidders is common knowledge at the bidding stage. In an informal auction, each individual bidder knows only the equilibrium entry strategy, but not the realized number of participating bidders at the time of choosing her bid. This auction format is meant to capture open selling procedures where interested bidders are invited to submit bids and the highest bidder wins and pays her own bid.

With correlated values, different types of bidders perceive the value of the good and their competitive environment differently. By affiliation, bidders with higher signals are intrinsically more optimistic about the value of the good. Similarly by affiliation, they believe that other bidders are more likely to be optimistic. If bids are increasing in signal, this means that bidders with high signals face more aggressive bidding from other bidders. Of course, these two different effects are present in auctions without entry costs. The second effect gains in importance once we add the entry costs to the model. If the cost is sunk before the auction takes place, losing to competing bidders becomes a more important consideration. We show that in any symmetric equilibrium of the auction models that we consider bidders with higher signals pay the entry cost with a higher probability than the lower types but in many cases lower types also enter. We show that uncertainty and more importantly different beliefs about the number of opponents give rise to non-monotonic bidding in the sense that participating bidders with high types sometimes lose to lower types at the auction stage.

We concentrate on symmetric equilibria of such models. By this solution concept, we emphasize settings where the set of potential bidders is not well known in advance. Examples of such cases include internet auctions and takeover bid contests. If the set of truly interested potential bidders is not known in advance, any mechanism based on eliciting all bidders' types fails if there are costs for contacting the potential bidders.

We consider two different types of entry costs. In our main model, potential bidders have already observed a signal on the value of the object when deciding whether to pay the entry cost. We call this the *interim entry* model. In the appendix, we also discuss the *ex ante entry* case where the cost is paid at an initial stage where all potential bidders are symmetrically informed. In both of these formats, bids are decided before the uncertainty on other potential bidders' participation decisions is resolved.

The analysis of such auctions is hard because in contrast to the case of formal auctions, it is possible that the game has no symmetric equilibria in monotone strategies.[1] Without monotonicity, it is not clear how one should proceed in the standard affiliated model with a continuum of signals for each bidder as in Milgrom and Weber (1982).

[1] Landsberger (2007) is an early paper showing that existence of monotone equilibria often fails in auctions with participation costs.

In order to make progress, we consider a finite mineral rights model where the unknown common value of the object is a binary random variable. The signals are drawn from a finite set and they are assumed to be independent across the potential bidders conditional on the true value of the object. In an informal auction, each bidder is uncertain about the number of competing bidders. With correlated signals, different types of bidders have different beliefs on the type profiles and realized numbers of their competitors. Two types of non-monotonicities emerge. Entry decisions may be non-monotonic in the sense that multiple types of bidders may enter with positive probability, and the bidding strategies may be non-monotone in the sense that a bid from a lower type wins over a bid of a higher type with positive probability.

One key feature of our equilibrium construction is that only a limited number of bidder types enter the auction in equilibrium. With a binary underlying state of the world, we can base our analysis on the bidders' expected payoffs conditional on the state. With affiliated types, the beliefs of the potential bidders on the state of the world are monotone in their types. Since we assume a large number of potential bidders, expected winnings at the auction stage must equal the entry cost in any equilibrium of the game. Suppose that the expected payoff at a fixed bid in the support of the equilibrium bid functions differs across the two states. Without loss of generality, assume that the bid generates a strictly higher payoff in state 1 than in state 0. In equilibrium, this bid can only be made by those bidders whose private signal assigns the highest probability to state 1. To see this, notice that if another bidder makes this bid, then her payoff must be at least equal to her entry cost. But by affiliation, the most optimistic bidder in her assessment on the probability of state 1 makes then a strictly positive profit contradicting our requirement of zero profits in equilibrium. We show that all the bid distributions that can emerge in a symmetric equilibrium of our model can be generated in a model where we consider only the most extreme types in the set of potential bidders, i.e. the two bidders with the most extreme beliefs on the two states.

For this reason, the main analysis in this paper is concentrated on the two-type case. We characterize the bidding equilibria of the informal first-price auction in this case. In the appendix, we show that our main results remain valid for the ex ante entry case as well. In that case, we must restrict our attention to a model with binary types since all bidder types are now present at the bidding stage of the game. Since we do not have analytical results on the bidding stage with multiple bidder types, this restriction on types is not without loss of generality, in contrast to the case with interim entry.

We start our analysis by solving the model with only two potential bidders. In this case, we show that the expected revenue to the seller from an informal first-price auction exceeds (at least weakly) the expected revenue from standard formal auction formats in the affiliated common values auction. This implies that in contrast to the linkage principle, the seller may be better off by withholding information from the bidders.[2]

Milgrom and Weber (1982) demonstrate the revenue superiority of the formal second-price auction over formal first-price auction for affiliated common value auctions. This result together with a set of results demonstrating the good revenue properties of public information disclosure are also known as the linkage principle. The key idea is that for a fixed own bid, any auction format that increases the linkage between own information and the perception of other players' bids increases the expected payment. To see how this principle fails in our informal auctions, consider an equilibrium in the second-price auction where low type bidders bid below the bid of the high types. By placing a bid between these two bids, a deviating bidder wins if and only if no high bidders participate in the auction. In this case, the payment is either the low bid if there is competition, or zero if the other bidder did not participate. By affiliation, it is more likely that no bidders with a low signal participate if the value of the object is high. But this means that the expected payment of the high type bidder is lower than the expected payment of the low bidder.

With more than two bidders, informal auctions may potentially possess multiple equilibria. If the winner of an auction is tied with other bidders with positive probability, information conditional on winning the object reflects the rationing amongst the highest bidders. This is not accounted for in the standard conditioning events of auctions with atomless bid distributions.[3] We show that in the informal first-price auction, symmetric equilibrium bidding strategies cannot have atoms in the interim entry case. This leads to a unique symmetric equilibrium outcome in terms of bid distributions in the overall game determining both entry and bids.

We model the game with many bidders as a Poisson game where the type of each potential player is positively correlated with the value of the object. The number of entrants of each type is drawn from a Poisson distribution with a parameter that depends on the true binary value of the object. An equilibrium of this game is a distribution of bids such that all entrants can cover their cost by their expected profit and no potential bidder of either type can make a positive profit by entering.

When comparing informal auctions to formal auctions, a second consideration emerges. The price paid in the formal auction is determined by the realized number of participating bidders. Whenever a bidder is the sole participant, she gets the object for free. For a formal auction with $n-1$ other participants, the equilibrium bid by the low bidders is equal to the value of the object conditional on n low signals. Due to affiliation, this payment is decreasing in n. For low values of

[2] With binary signals, formal first-price and second-price auctions result in the same expected revenue, the same expected bidder rents and hence the same entry decisions.

[3] Pesendorfer and Swinkels (1997) point out this possibility in the case of formal auctions and Lauermann and Wolinsky (2015) discuss this issue in a first-price auction with an unknown number of bidders.

the entry cost, the probability of the event that no other bidders are present vanishes. Since a high type bidder assigns a higher probability to lower n, we see that the expected payment of the bidder is positively linked with the type and the usual linkage principle applies. This explains our finding that formal second-price auction dominates informal first-price auction if the entry cost is very low. But when the entry costs are not too small, we show that the expected revenue ranking from the two-bidder case carries over to the Poisson game with endogenous entry.

We get our strongest results when the entry cost is relatively high (and the expected number of entrants is relatively low), and the affiliation in the signals is strong. In this case, zero bid is in the support of the equilibrium bidding strategies of both types of players in the informal first price auction. With atomless bidding strategies this means that the payoff of each type of bidder coincides with the value of the good conditional on being the only entrant. This private benefit is also the maximum social benefit from inducing additional entry when restricted to symmetric strategies. We conclude that the symmetric equilibrium entry rates maximize social welfare in the class of symmetric entry strategies. Since we have large numbers of potential bidders, the expected payoff to bidders net of entry costs must be zero and as a result the seller receives the maximal symmetric surplus as her expected revenue in this class of auctions. For these parameter values we see then that the informal first price auction is the revenue maximizing mechanism in the class of symmetric mechanisms.

1.1 Related Literature

Endogenous entry into auctions has been modeled in two separate frameworks. In the first, entry decisions are taken at an ex ante stage where all bidders are identical. Potential bidders learn their private information only upon paying the entry cost. Hence these models can be though of as games with endogenous information acquisition.[4] French and McCormick (1984) gives the first analysis of an auction with an entry fee in the IPV case. Harstad (1990) and Levin and Smith (1994) analyze the affiliated interdependent values case. These papers show that due to business stealing, entry is excessive relative to social optimum. They also show that second-price auctions dominate the first-price auction in terms of expected revenue. All of these papers proceed under the assumption that the number of entering bidders is known at the moment when bids are submitted. Our informal auctions are thus not covered at all in these papers.

In the other strand, bidders decide on entry only after knowing their own signals. Samuleson

[4]The equilibrium determination of information accuracy in common values auctions started with Matthews (1977) and Matthews (1984) and Persico (2000) extended this line of research to revenue comparisons for different auction formats. Since the number of bidders and equilibrium information acquisition decisions are deterministic, equilibrium bidding in these papers is still as in standard affiliated auctions models.

(1985) and Stegeman (1996) are early papers in the independent private values setting where this question has been taken up. Due to revenue equivalence, comparisons across auction formats are not very interesting. To the best of our knowledge, common values auctions have not been analyzed in this setting.[5]

Finally some recent papers have analyzed common values auctions with some similarities to our paper. Pekec and Tsetlin (2008) provides an example where informal first-price auction results in a higher expected revenue than an informal second-price auction. The distribution of the bidders in that paper is somewhat extreme and not derived from entry decisions. Lauermann and Wolinsky (2017) and Lauermann and Wolinsky (2019) analyze first-price auctions where an informed seller chooses the number of bidders to invite to an auction. The bidders do not observe how many others were invited and hence the bidding stage analysis is as in our model with an exogenous entry rate. These papers do not compare revenues across different auction formats and since the distribution of entering bidders results from an optimal invitation decision by the seller, the analysis is quite different from our paper. Atakan and Ekmekci (2014) consider a common value auction where the winner in the auction has to take an additional action after winning the auction. This leads to a non-monotonicity in the value of winning the auction that has some resemblance to the forces in our model that lead to non-monotonic entry (i.e. bidders with both types of signals enter with positive probability).

Since we concentrate on symmetric equilibria of a game with a large number of potential entrants, our model has some similarities to the urn-ball models of matching. Similar to those models, our insistence on symmetric equilibria can be seen as a way of capturing a friction in the market that precludes coordinated asymmetric decisions. A recent example of such models is Kim and Kircher (2015) that studies matching with private values uncertainty. This approach has also been used in Jehiel and Lamy (2015) a procurement auction setting with private (asymmetric) values. These models have not covered the case of common values to the best of our knowledge.

The paper is structured as follows. In Section 2, we introduce the main ideas of the paper in the simplest possible setting with only two potential bidders. Section 3 presents the informal first-price auction and shows that in equilibrium at most two types enter. The section also analyzes equilibrium bidding in a model with an exogenous random number of bidders with binary types. Section 4 analyzes equilibrium entry decisions and shows that equilibrium outcomes are essentially unique. Section 5 compares the expected revenues across different auction formats and Section 6 discusses the modeling assumptions in this paper. All proofs of the results are in Appendix B.

[5]Bhattacharya, Roberts, and Sweeting (2014) and Sweeting and Bhattacharya (2015) analyze IPV models with selective entry where entry decisions are conditioned on private information on the true valuation.

2 Two-Player Model

We start with the analysis of the model with two types of bidders and only two potential bidders. By doing this, we can introduce the main ideas in the paper with minimal notation and without analytical complications that are unavoidable with many bidders.

Two potential bidders with a signal $\widetilde{\theta} \in \{l, h\}$ must decide whether to pay an entry cost c to participate in an auction for a single indivisible good. The value of the good $v(\omega)$ is binary $\widetilde{\omega} \in \{0, 1\}$ with $v(1) > v(0) > c$ and the signals are i.i.d. conditional on v. By affiliation, let

$$\alpha := \Pr\{\widetilde{\theta} = h \,|\widetilde{\omega} = 1\} \;>\; \Pr\{\widetilde{\theta} = h \,|\widetilde{\omega} = 0\} =: \beta.$$

Let $q = \Pr\{\widetilde{\omega} = 1\}$ be the prior probability on the value of the object.

The bidders are risk-neutral and they maximize their payoff at the auction stage net of the entry cost.

2.1 Symmetric Social Optimum

In this subsection, we analyze a social planner's problem to obtain a useful benchmark for symmetric equilibria of various auction formats. Hence, we restrict the planners' feasible strategies to be also symmetric. The planner chooses entry probabilities (π_l, π_h) for players with signals l and h respectively to

$$\begin{aligned} \max_{0 \le \pi, \pi_h \le 1} & q[\left(1 - (1 - \alpha\pi_h - (1-\alpha)\pi_l)^2\right) v(1) - 2c(\alpha\pi_h + (1-\alpha)\pi_l)] \\ & + (1-q)\left[\left(1 - (1 - \beta\pi_h - (1-\beta)\pi_l)^2\right) v(0) - 2c(\beta\pi_h + (1-\beta)\pi_l)\right]. \end{aligned} \quad (1)$$

The two first-order conditions for an interior solution to this concave quadratic problem can be written as:

$$q_h (1 - \alpha\pi_h - (1-\alpha)\pi_l) v(1) + (1-q_h)(1 - \beta\pi_h - (1-\beta)\pi_l) v(0) = c, \quad (2)$$

$$q_l (1 - \alpha\pi_h - (1-\alpha)\pi_l) v(1) + (1-q_l)(1 - \beta\pi_h - (1-\beta)\pi_l) v(0) = c, \quad (3)$$

where q_θ is the posterior of type θ bidder:

$$\begin{aligned} q_h &= \frac{\alpha q}{\alpha q + \beta(1-q)} > \\ q_l &= \frac{(1-\alpha) q}{(1-\alpha) q + (1-\beta)(1-q)}. \end{aligned}$$

The two lines state that the cost of adding a player of each type must equal the benefit. The benefit is realized only if the other bidder did not participate. Notice also that an immediate implication of this first-order condition is that the benefits be equalized also across states (since $q_h > q_l$). Since $v(0) < v(1)$, we see from this that $\pi_h^{opt} > \pi_l^{opt}$ in the solutions to these equations.

For completeness, let us note that the first order condition is valid only for $0 < \pi_l^{opt} < \pi_h^{opt} < 1$. The first inequality $0 < \pi_l^{opt}$ holds if

$$\frac{\alpha}{\beta} > \frac{(v(1) - c)/v(1)}{(v(0) - c)/v(0)}.$$

The second inequality $\pi_h^{opt} < 1$ holds if

$$\frac{\alpha}{\beta} > \frac{v(1)}{v(0)}.$$

Finally, note that if $\pi_l^{opt} = 0$, then $\pi_h^{opt} > 0$ if $q_h v(1) + (1 - q_h) v(0) > c$.

We see that we get interior solutions if entry costs are small and if the ratio of the valuations is not too large. The reason why the planner wants both types of players to participate is that the types of the players are correlated with the state of the world. Ideally the planner would tailor the entry probabilities to the state of the world, but this information is not available to her at the beginning of the game. By inducing entry from both types of players, the planner can balance the benefits of entry across the two states.

In the case with a large number of potential bidders, the equivalent of the second inequality is not binding since expected number of entrants is always bounded by $\frac{v_1}{c}$ in any solution (optimal or equilibrium) of the model.

2.2 Informal First-Price Auction

We analyze next the symmetric Bayes-Nash equilibria of the game where each potential bidder decides simultaneously whether to enter and what to bid conditional on entry. Formally, the strategy of player each player maps her type θ to a probability of participating π_θ and a bid distribution $F_\theta(\cdot)$. We use $\lambda_{\omega,\theta}$ to denote the probability with which a player of type θ enters in state ω if the players use the symmetric entry strategies (π_l, π_h) :

$$\begin{aligned} \lambda_{1,h} &= \alpha\pi_h, \ \lambda_{0,h} = \beta\pi_h, \\ \lambda_{1,l} &= (1 - \alpha)\pi_l, \ \lambda_{0,l} = (1 - \beta)\pi_l . \end{aligned}$$

With this notation, we can write the expected payoff $U_\theta(p)$ at the bidding stage to type θ from bidding p (when other players use strategy $(\pi_l, \pi_h), (F_l(\cdot), F_h(\cdot)))$ as:

$$\begin{aligned} U_\theta(p) \quad : \quad &= q_\theta \left(1 - \lambda_{1,h}(1 - F_h(p)) - \lambda_{1,l}(1 - F_l(p))\right)(v(1) - p) \\ &\quad + (1 - q_\theta)\left(1 - \lambda_{0,h}(1 - F_h(p)) - \lambda_{0,l}(1 - F_l(p))\right)(v(0) - p) \\ = \quad &: q_\theta R_1(p) + (1 - q_\theta) R_0(p), \end{aligned}$$

where $R_\omega(p)$ denote the expected rent in state ω from bid p. In a symmetric Bayes-Nash equilibrium, for all p in the support of $F_\theta(\cdot)$, p maximizes $U_\theta(p)$.

We denote the highest bid in the union of the supports of the two bid distributions by $p^{\max}$. Our next proposition provides a characterization of the unique bidding equilibrium for the bidding stage of the informal first-price auctions with $0 < \pi_l \leq \pi_h < 1$.[6] We stress here that this characterization does not hinge on (π_l, π_h) being an equilibrium entry pair. This is important for our discussion of the alternative ex ante entry model in Appendix A where $\pi_h = \pi_l$ by construction since players do not know their types when choosing entry.

Proposition 1 *Assume exogenous entry probabilities* $0 < \pi_l \leq \pi_h < 1$.*The informal first-price auction with two potential bidders has a unique symmetric equilibrium, where both types of bidders use atomless mixed strategies. The supports of the two bid distributions* $F_\theta(\cdot)$ *for* $\theta \in \{l, h\}$ *satisfy*

1. $0 \in \mathit{supp}F_l(\cdot), p^{\max} \in \mathit{supp}F_h(\cdot),\ \mathit{supp}F_l(\cdot) \cup \mathit{supp}F_h(\cdot) = [0, p^{\max}]$,

2. *Either* $0 \in \mathit{supp}F_h(\cdot)$ *or there exists a* $p' > 0$ *such that* $\mathit{supp}F_l(\cdot) = [0, p']$ *and* $\mathit{supp}F_h(\cdot) = [p', p^{\max}]$.

The first property to notice is that the bid distributions contain no atoms. While this is a standard feature of auction models with a known number of bidders, the result is not true in general in models with a random number of participating bidders. With only two bidders, there is a single event that can result in a tied winning bid: both bidders submit the same bid. In this case, winning the object conveys no additional information and because of this, the usual argument that implies atomless distributions holds. In the next section, we show that with many potential bidders, the interim entry model has a unique equilibrium and it is in atomless strategies. For the case with ex ante entry, this is not true.

The second key feature is that the supports of the distribution take very specific forms. Low type bidders always have the zero bid in their equilibrium bid support. This means that they earn a positive payoff only if the other bidder does not participate. But the expected payoff from this bid is exactly the player's contribution to the social surplus. If zero bids are also in the support of the

[6] We deal with the corner solutions separeately in the subsection on revenue comparisons.

high type bidder, then both types of players earn as their equilibrium payoffs exactly their marginal contribution. In the other possible case, we know that we can compute the expected payoff to the high type bidder by calculating her payoff at the highest bid that the low type bidders make in equilibrium. These observations turn out to be useful for the revenue comparisons below.

2.3 Formal Auctions

As a benchmark for comparison, consider formal (standard) auctions, where the bidders know the number of other bidders at the time of placing their bids. We have shown in Chi, Murto, and Välimäki (2019) that both first- and second-price auctions have a unique symmetric equilibrium in the current environment.

In formal auctions, it is naturally optimal for any bidder to bid zero if there are no other bidders. Whenever there are at least two bidders present, the unique equilibrium in the first-price auction results in the bid

$$b_l = \mathbb{E}\left(v \,|\theta_1 = \theta_2 = l\,\right)$$

for the low type.High type bidders mix on an interval $[b_l, p']$, $p' > b_l$. The low type bidder gets a strictly positive payoff equal to her expected value of the object only if she is the only bidder present. Notice that this is also the social value of entry. The high type bidder earns an information rent on top of this social value since at bid b_l she wins with positive probability and earns a strictly positive expected payoff equal to the difference between the expected value of the object based on her high signal and that based on a low signal.

The formal second price auction has also a unique symmetric equilibrium where $\theta \in \{l, h\}$ submits the bid

$$b_\theta = \mathbb{E}\left(v\left|\widetilde{\theta}_1 = \widetilde{\theta}_2 = \theta\right.\right).$$

The high type gets in this case exactly the same information rent as in the first-price auction. We can then conclude that formal first-price auction and formal second-price auction result in the same payoffs to both types. Furthermore we see that the expected payoff would be the same also in the informal second-price auction since the tying event is uniquely determined. Hence it is sufficient to compare the expected revenues between the informal first-price auction and the formal second-price auction.

2.4 Revenue Comparisons with Exogenous Entry

With two potential bidders, we get an unambiguous revenue ranking for the auction formats that we consider: the informal first-price auction is superior in terms of expected revenue to the other formats. Note that in a model with common values, allocation is always efficient so that an increase in revenue is at the expense of rents to the bidders. Hence, higher expected revenue is equivalent to lower bidder rents.

Proposition 2 *With two potential bidders and exogenous entry probabilities $0 < \pi_l \leq \pi_h < 1$, the informal first-price auction generates a higher expected revenue than the formal auctions. Equivalently, the expected rents to the bidders are lower in the informal first-price auction than in the formal auctions.*

The key step in the proof of the proposition compares the expected payments of the high type bidders across the informal first-price auction and the formal second-price auction. Even though the types themselves are affiliated in the usual sense, the fact that low types enter with a lower probability in a high state generates a different type of dependence between the types of participating bidders resulting in different expected payments. If the model had more signals, then affiliation in the types would add a counteracting effect improving the revenue performance of the formal second-price auction relative to the first-price auction.

2.5 Interim Entry Equilibrium with Two Potential Bidders

We conclude this section with the analysis of entry decisions. This specification is particularly relevant for cases where differential selection of the bidders plays a key role. The interesting situation is the one where both types enter with positive probability. Any such equilibrium trades off two forces in a way that makes entry viable for both types. High types are more optimistic about the value of the object. Low types on the other hand, find a low level of competition more likely. The differences in the equilibrium outcomes of different auction formats result from the different ways in which they balance this trade-off. This feeds directly into different revenue properties of the auctions as we show in this section.

Our first observation relates equilibrium social surplus to the expected revenue of the seller. If both types are indifferent between entering or not, i.e. is entry is with interior probabilities $0 < \pi_l \leq \pi_h < 1$, then the bidders earn no expected rent in the game. In other words, their expected payoff in equilibrium is fully dissipated by the entry cost c. In our model with quasilinear payoffs, this implies that the seller's expected revenue coincides with the social surplus generated. With this observation, we can conduct our revenue comparisons in terms of the social surplus generated in the symmetric equilibrium of the game, as in Levin and Smith (1994).

We can view the first order conditions (2) and (3) as the planner's reaction curves. For each fixed value of π_l, (2) gives the socially optimal level $\pi_h^*(\pi_l)$ and for each level of π_h, (3) gives the socially optimal level $\pi_l^*(\pi_h)$.

The private costs of a potential entrant coincide with the social cost. Since the private benefit is at least equal to the social benefit, we see that our auction formats generate excessive entry relative to social optimum.[7] By the characterization result in the previous subsection, low types always have the zero bid in the support of their bid distribution. This means that their entry is conditionally efficient given the entry rate of the high types. This implies, in particular, that symmetric entry equilibria in all of the auction formats that we consider lie on the planner's reaction curve $\pi_l^*(\pi_h)$.

Equilibrium entry $\pi_l^*(\pi_h)$ probability of the low types is decreasing and linear in π_h. This observation together with the concavity of the social objective function implies that the auction format that generates less entry by high types generates a higher social surplus. Whenever we have interior entry probabilities, a comparison of the entry rates of the high types then also gives us a revenue ranking for the auction formats.

Consider next the equilibrium payoff to the high types at the bidding stage of the informal first-price auction. We can show that for each level of π_l, there is a unique level of $\pi_h^{IF}(\pi_l)$ such that

$$V_h\left(\pi_h^{IF}(\pi_l), \pi_l\right) := \max_p U_h^\lambda(p) = c,$$

where $U_h^\lambda(p)$ is the equilibrium payoff to the high type in the informal first-price auction parametrized by entry probabilities

$$\begin{aligned}\lambda_{1,h} &= \alpha\pi_h^{IF}(\pi_l),\ \lambda_{0,h} = \beta\pi_h^{IF}(\pi_l),\\ \lambda_{1,l} &= (1-\alpha)\pi_l,\ \lambda_{0,l} = (1-\beta)\pi_l.\end{aligned}$$

Furthermore, we can show that $\pi_h^{IF}(\pi_l)$ is continuous in π_l and that for all π_l, $\pi_h^{IF}(\pi_l) \geq \pi_h^*(\pi_l)$.

Let $\left(\pi_h^{IF}, \pi_l^{IF}\right)$ solve

$$\begin{aligned}\pi_h^{IF}\left(\pi_l^{IF}\right) &= \pi_h^{IF},\\ \pi_l^*\left(\pi_h^{IF}\right) &= \pi_l^{IF}.\end{aligned}$$

By continuity and $\pi_h^{IF}(\pi_l) \geq \pi_h^*(\pi_l)$, we know that at any equilibrium solution, $\pi_h^{IF} \geq \pi_h^{opt}$ and $\pi_l^{IF} \leq \pi_l^{opt}$. Using the properties of the bidding equilibrium, we can also show that $\left(\pi_h^{IF}, \pi_l^{IF}\right)$ is unique. We omit the details here since the reasoning is completely analogoue to the case with Poisson entry proved in full in Proposition 10.

Recall from Proposition 1 that it is possible that zero is also in the support of the bids made by the high type in the bidding equilibrium. In this case, the bidders earn exactly by their social contribution at the auction stage and $\pi_h^{IF}(\pi_l) = \pi_h^*(\pi_l)$. As a result, the entry rates then socially

[7] By bidding zero in either of the auction formats, both types of bidders can secure at least the social value (i.e. the value of the object in the event that there are no other bidders).

optimal, $\pi_h^{IF} = \pi_h^{opt}$, $\pi_l^{IF} = \pi_l^{opt}$, and the revenue coincides with the optimal social surplus in the planner's problem. This implies that there can be no other symmetric game forms with higher revenues where entry decisions are taken based on own types only. Of course in a correlated model, one could improve on the performance of the auction if entry could be conditioned on the vector of reported types. In the spirit of costly participation, any communication prior to deciding entry should also have an associated cost, which we take here to be prohibitively high.

We can derive a similar equilibrium reaction curve for the rate of entry $\pi_h^F(\pi_l)$ in the formal auctions that keeps high types indifferent between entering and not entering. Proposition 2 shows that with fixed entry rates, the high types earn more in formal auctions than in the informal first-price auction. When entry rates are endogenous, this must be compensated by a higher entry rate in the formal auction so that $\pi_h^F(\pi_l) \geq \pi_h^{IF}(\pi_l) \geq \pi_h^*(\pi_l)$.

We can summarize our discussion in the following proposition showing that the revenue ranking from the model with exogenous entry holds also with equilibrium entry.

Proposition 3 *With interim entry decisions, informal first-price auction generates a higher expected revenue than the formal auctions.*

We end this section with a few words regarding the model with ex ante entry. We defer the formal analysis of that model to Appendix A. With two potential entrants, very little changes relative to the model with interim entry. Since bidders do not know their types at the moment of choosing entry, there is a single entry rate $\pi = \pi_l = \pi_h$. Since Propositions 1 and 2 cover this case as well, we conclude that the informal first-price auction results in smaller expected equilibrium payoffs to the bidders than formal auctions. These payoffs always exceed (weakly) the players' marginal contribution to social surplus, and as a result, entry rate is distorted upwards relative to the symmetric planner optimum in the concave planner's problem. Since the informal first-price auction results in smaller distortions in the entry rate, the conclusion from the interim case remains valid and the informal first-price auction dominates in terms of expected revenue.

3 Informal First-Price Auction with Poisson Entry

In our main model, we assume that the number of entrants is a Poisson random variable with an *endogenously* given parameter. Since the seminal work of Myerson on games with uncertain numbers of participants, the Poisson game model has been widely used in models of information economics.[8] It is a tractable model that allows the random number of participating bidders to be

[8]For a discussion of the Poisson entry model in the context of a procurement auction with private values, see Jehiel and Lamy (2015).

correlated with the state of the world (the true value of the object). As in the previous section with two potential participants, this correlation can be rationalized by entry decisions based on affiliated signals.

To see how one arrives naturally at the Poisson model, assume for the moment that there are N potential entrants that observe signals $\widetilde{\theta} \in \{1, ..., M\}$ with a conditional distribution

$$\alpha_{\omega,m} =: \Pr\{\widetilde{\theta} = m \,|\widetilde{\omega} = \omega\} \text{ for } m \in \{1, ..., M\},$$

where state is $\omega \in \{0, 1\}$. We label the signals so that they satisfy strict monotone likelihood ratio property: $\frac{\alpha_{1,m}}{\alpha_{0,m}}$ is increasing in m. If each entrant with signal m enters with probability $\pi_{m,N}$, then the number of high type entrants is a binomial random variable with parameters $(\alpha_{1,m}\pi_{m,N}, N)$ and $(\alpha_{0,m}\pi_{m,N}, N)$ in states $\omega = 1$ and $\omega = 0$, respectively. Keeping the expected number of entrants $\alpha_{\omega,m}\pi_{m,N} \cdot N \to \alpha_{\omega,m}\pi_m$ constant, let N increase towards infinity and consider the binomial variables $\text{Bin}(\alpha_{\omega,m}\pi_{m,N}, N)$. The limiting random number N_ω^m of entrants of type m in state ω has then a Poisson distribution with parameter $\lambda_{\omega,m} := \alpha_{\omega,m}\pi_m$. The random variables N_ω^m are furthermore independent. As before, we use the notation N^m to denote the total random number of participating bidders of type m.

Motivated by this limiting argument, we model the entry game directly as a Poisson game where π_m are endogenously determined parameters. Each potential bidder perceives the number of other participants of type m to be given by a Poisson random variable N_ω^m with parameter $\lambda_{\omega,m} = \alpha_{\omega,m}\pi_m$ that depends on π_m and the distribution of the signal $\widetilde{\theta}$. By a symmetric equilibrium, we mean a pair (b, π) where $b : \{1, ..., M\} \to \Delta(\mathbb{R}_+)$ is the equilibrium bidding strategy for participating bidders and $\pi : \{1, ..., M\} \to \mathbb{R}_+$ is the equilibrium entry rate. The equilibrium condition for entry rate $\pi = (\pi_1, ..., \pi_M)$ is that given (b, π), no potential bidder has a higher expected payoff than c at the auction stage and if $p \in \text{supp} F_m(\cdot)$ for some m, then her expected payoff in the auction is c.

3.1 Symmetric Planner's Optimum

We start again with the symmetric surplus maximizing benchmark where a social planner chooses $\pi = (\pi_1, ..., \pi_M)$ to maximize the expected gain from allocating the object net of the expected entry cost. The planner's objective is to

$$\max_{\pi \geq 0} W(\pi),$$

where $W(\pi)$ is the expected total surplus with a given entry profile:

$$W(\pi) = q[v(1)\left(1 - e^{-\Sigma_m \alpha_{1,m}\pi_m}\right) - c\Sigma_m \alpha_{1,m}\pi_m]$$
$$+ (1-q)[v(0)\left(1 - e^{-\Sigma_m \alpha_{0,m}\pi_m}\right) - c\Sigma_m \alpha_{0,m}\pi_m].$$

The first order conditions for interior solutions to this concave problem are given by:

$$q_m v(1) e^{-\Sigma_m \alpha_{1,m}\pi_m} + (1-q_m) v(0) e^{-\Sigma_m \alpha_{0,m}\pi_m} = c,$$

for all m. Its unique solution is given by:

$$\begin{aligned} v(1) e^{-\Sigma_m \alpha_{1,m}\pi_m} &= c, \\ v(0) e^{-\Sigma_m \alpha_{0,m}\pi_m} &= c. \end{aligned} \tag{4}$$

Taking logarithms, we have

$$\begin{aligned} \Sigma_m \alpha_{1,m}\pi_m &= \log\left(\frac{v(1)}{c}\right), \\ \Sigma_m \alpha_{0,m}\pi_m &= \log\left(\frac{v(0)}{c}\right). \end{aligned}$$

Since we have two linear equations in M variables, the solutions to this pair of equations are not unique. By affiliation, we have $\frac{\alpha_{1,M}}{\alpha_{0,M}} \geq \frac{\alpha_{1,m}}{\alpha_{0,m}} \geq \frac{\alpha_{1,1}}{\alpha_{0,1}}$. Hence we have a positive solution for $\pi = (\pi_1, ..., \pi_M)$ only if

$$\begin{aligned} \alpha_{1,1}\pi_1 + \alpha_{1,M}\pi_M &= \log\left(\frac{v(1)}{c}\right), \\ \alpha_{0,1}\pi_1 + \alpha_{0,M}\pi_M &= \log\left(\frac{v(0)}{c}\right), \end{aligned}$$

has a positive solution. Again by affiliation, we see that this is the case only if

$$\alpha_{0,M} \log\left(\frac{v(1)}{c}\right) < \alpha_{1,M} \log\left(\frac{v(0)}{c}\right).$$

If this condition is satisfied, a solution $(\pi_1, 0, ..., 0, \pi_M)$ with $\pi_M > \pi_1 > 0$ to this system exists. Obviously other solutions to this problem exist, but the aggregate amount of entry is determinate for both states across all such solutions.

We can compute a threshold $\overline{c}$ such that positive entry by multiple types takes place if the entry cost is below $\overline{c}$:

$$\overline{c} = e^{\frac{\alpha_{1,M}\log(v(0)) - \alpha_{0,M}\log(v(1))}{\alpha_{1,M} - \alpha_{0,M}}} > 0.$$

For $c \in (0, \overline{c})$ we have an interior solution with $\pi_M^{opt} > \pi_1^{opt} > 0$.

When $c \geq \overline{c}$, we have a corner solution where $\pi_m^{opt} = 0$ for all $m < M$. In that case, the optimal entry rate for the θ_M types is solved from

$$q_M v(1) e^{-\alpha_{1,M}\pi_M} + (1-q_M) v(0) e^{-\alpha_{0,M}\pi_M} = c.$$

This gives an interior solution $\pi_M^{opt} > 0$ as long as $c < \bar{\bar{c}}$, where

$$\bar{\bar{c}} = q_h v(1) + (1 - q_h) v(0).$$

To summarize, for low entry cost $c < \bar{c}$ the socially optimal entry profile features the two extreme types entering with positive rate, for intermediate entry cost $c \in [\bar{c}, \bar{\bar{c}})$ only highest type enters with positive rate, and for high entry cost $c \geq \bar{\bar{c}}$ there is no entry. Note that the threshold $\bar{\bar{c}}$ is the expected value of the object for a player that has observed the highest signal. This is intuitive: as long as entry cost c is below the expected value of the object for the most optimistic potential entrant, it is socially optimal to have at least some entry.

3.2 No Pooling with Interim Entry

In any interim entry equilibrium, entrants earn an expected profit of c in the bidding stage. Hence for all bids p in the union of the bid supports of all bidders, the equilibrium payoff of any bidder submitting p must be c and the payoff for any other type cannot exceed c. Otherwise we would have a contradiction to interior entry probabilities.

With a random number of bidders, the analysis of pooling bids (i.e. bids that at least one of the types chooses with a strictly positive probability) is more delicate than in the case of a fixed number of bidders (or with two potential participants). With uniform tie-breaking for highest bids, a bidder submitting a pooling bid is more likely to win if the number of tying bids is small. The number of tying bids contains information on the realization of (N^h, N^l). Since N_0^θ and N_1^θ have different distributions, this information is in turn informative on the state of the world and hence on the value of the object. The additional information contained in the event of winning the auction must be accounted for when calculating the optimal bid. We call this effect the rationing effect of winning.

As observed by Pesendorfer and Swinkels (1997), it is not possible to have pooling bids that are made only by the highest types. If they were the only bidders to pool on a bid p, then the rationing effect would be negative: by bidding p, a win is more likely when there are few tied bidders. But if only high type bidders bid p, then winning with a tied bid decreases the posterior on $\{\omega = 1\}$ and the value of the object conditional on winning is lower than the value conditional on the event that the bidder is tied for the highest bid. By bidding $p + \varepsilon$, for a small enough ε, the bidder wins in the event of a tied bid without any rationing and makes a positive gain from the deviation. As a result, pooling by high types only is not possible.

To rule out other types of pooling bids, we note first that there cannot be pooling at any $p < v(0)$. This is because for such a low bid, winning is profitable in both states and hence a small upward deviation would be strictly profitable. On the other hand, with interim entry only the highest type M can bid above $v(0)$. Winning with a bid $p \geq v(0)$ is profitable only if $\omega = 1$. By affiliation, $\Pr\{\omega = 1 \,|m\}$ is increasing in m. Therefore, if m gets in expectation c by bidding $p \geq v(0)$, then type M gets strictly more than c by bidding p, which is not compatible with interim entry equilibrium. Since it is not possible to have pooling bids that are made only by high types, pooling is not possible for any $p \geq v(0)$ either. We have hence proved:

Proposition 4 *There are no atoms in the bidding distribution b of a symmetric interim entry equilibrium (b, π).*

3.3 Entry by Extreme Types

We consider next the types of bidders that can enter in eqilibria with atomless bidding strategies at the auction stage. As in Section 2, denote by $R_\omega(p)$ the expected rent at the auction stage in state ω from bid p:

$$R_\omega(p) = (v(\omega) - p) \prod_{m=1}^{M} e^{-\lambda_{\omega,m}(1-F_m(p))}.$$

For each p, there are three possibilities: either i) $R_1(p) = R_0(p)$, ii) $R_1(p) > R_0(p)$, or iii) $R_1(p) < R_0(p)$. All types of bidders are indifferent beween entering and submitting a bid p in case i). In case ii), only type θ_M can make bid p in equilibrium with interior entry probabilities. If $m < M$ makes the bid, she must earn at least c. Since her expected payoff at the bidding stage is $R_0(p) + q_m(R_1(p) - R_0(p))$, we see that type is not indifferent contradicting interior entry. In case iii), we have similarly that only type 1 can enter in such an equilibrium.

The equilibrium construction in Proposition 6 shows that the bid distribution is fully pinned down by the parameters of the bidders that bid in cases ii) and iii). The only remaining indeterminacy concerns the exact composition of bidders that make bids where case i) holds. But the requirement that the expected payoffs be equalized across the two states determines the aggregate distributions of bids for each state in a manner completely analogous to the payoff equalization in the planner's problem across the two states. As in the planner's case, these aggregate distributions can be generated with positive entry by types 1 and M. We formalize this discussion in the following proposition,

Proposition 5 *Let $\left(\bar{b}, (\overline{\pi}_1, \overline{\pi}_M)\right)$ denote an equilibrium of a reduced model, where only types $\{1, M\}$ exist. Then $\left(\bar{b}, (\overline{\pi}_1, 0, ..., 0, \overline{\pi}_M)\right)$ remains an equilibrium of the full model that allows entry by all types $\{1, ..., M\}$. Moreover, if there is another equilibrium (b, π) in the full model, this equilibrium is equivalent to an equilibrium of the reduced model in terms of induced probability distribution of bids and revenues.*

Based on these two results, we restrict our attention for the rest of this section to atomless bidding equilibria of a model with binary bidder types $\theta \in \{l, h\}$, where l corresponds to type 1 and h corresponds to type M.

3.4 Bidding Equilibrium with Exogenous Entry by Extreme Types

We start by analyzing the bidding stage in the case where the number of entrants is determined by exogenously given entry rates π_h and π_l. A bidder with signal $\theta \in \{l, h\}$ chooses her optimal bid in informal auctions depending on her updated probability on the state $q_\theta := \Pr\{\omega = 1 \,|\theta\}$ and her conditional distribution on the realized number of (other) bidders with signal θ. The number of bidders with signal θ in state ω is a Poisson random variable N_ω^θ, where the parameter $\lambda_{\omega,\theta}$ depends on π_θ through:

$$\begin{aligned} \lambda_{1,h} &= \alpha\pi_h, \ \lambda_{0,h} = \beta\pi_h, \\ \lambda_{1,l} &= (1-\alpha)\,\pi_l, \ \lambda_{0,l} = (1-\beta)\,\pi_l\,, \end{aligned}$$

where $\alpha > \beta$ implies that $\lambda_{1,h} > \lambda_{0,h}$ and $\lambda_{1,l} < \lambda_{0,l}$. While the entry probabilities π_h and π_l are treated here as exogenous, they will be endogenized in the next section. Since we consider here the bidding behavior of an individual bidder, we use N_ω^θ to denote the number of bidders *excluding the bidder under consideration*.[9]

3.4.1 Symmetric Equilibrium in Atomless Strategies

Consider equilibrium bidding in the informal first price auction with an exogenously given distribution of bidders. The bidder submitting the highest bid $p \geq 0$ wins the object and pays her bid, any ties are broken symmetrically between highest bidders, and bidders that do not win make no payments. We show in this section that this model has a unique symmetric equilibrium in atomless strategies.

Based on q_θ and the entry rates $\boldsymbol{\lambda} = \{\lambda_{\omega,\theta}\}$, each bidder of type θ computes her posterior beliefs on the numbers of other participants and forms expectations about their bids. As in the Section 2, we denote by $F_\theta\,(p)$ the (continuous) c.d.f. of bids below level p by any bidder with signal θ. Since the nubers of participating bidders are drawn from independent Poisson distributions and since the randomizations over bids are independent across bidders, the probability of winning in state ω at

[9]Note that in a Poisson model, an individual bidder perceives the number of other bidders distributed according to the same distribution as an outsider sees the number of all bidders in the game (conditional on state).

bid p is given by:

$$\begin{aligned}
&\sum_{n^h \geq 0} \sum_{n^l \geq 0} \Pr\left(N^h = n^h \left|\omega\right.\right) F_h(p)^{n^h} \Pr\left(N^l = n^l \left|\omega\right.\right) F_l(p)^{n^l} \\
= &\sum_{n^h \geq 0} \frac{(\lambda_{\omega,l})^{n^h} e^{-\lambda_{\omega,h}}}{n^h!} F_h(p)^{n^h} \sum_{n^l \geq 0} \frac{(\lambda_{\omega,l})^{n^l} e^{-\lambda_{\omega,l}}}{n^l!} F_l(p)^{n^l} \\
= &\ e^{-\lambda_{\omega,h}(1-F_h(p))} e^{-\lambda_{\omega,l}(1-F_l(p))}.
\end{aligned}$$

Using this winning probability, we can compute the expected payoff from bid p to a bidder of type θ when the other players bid according to the profile $\mathbf{F} = (F_h(\cdot), F_l(\cdot))$ as follows:

$$\begin{aligned}
U_\theta^{\boldsymbol{\lambda}}(p) &= q_\theta e^{-\lambda_{1,h}(1-F_h(p))} e^{-\lambda_{1,l}(1-F_l(p))} (v(1) - p) \\
&\quad + (1 - q_\theta) e^{-\lambda_{0,h}(1-F_h(p))} e^{-\lambda_{0,l}(1-F_l(p))} (v(0) - p).
\end{aligned}$$

A symmetric bidding equilibrium for entry rates $\boldsymbol{\lambda}$ is a bid profile $\mathbf{F}^{\boldsymbol{\lambda}}$ such that for all $p \in \text{supp} F_\theta^{\boldsymbol{\lambda}}(\cdot)$, p maximizes $U_\theta^{\boldsymbol{\lambda}}(p)$. For the remainder of this section, we fix $\boldsymbol{\lambda}$ and omit it in the notation for the bid distributions.

The main result of this section characterizes atomless symmetric equilibrium bid distributions. This characterization is remarkably similar to Proposition 1 in the two-bidder case. If anything, the proof of the result is simpler than in the two-bidder case and the result is sharper in the sense that it gives a necessary and sufficient condition for the two qualitatively different types of equilibria.

Proposition 6 *The informal first-price auction has a unique symmetric equilibrium in atomless bidding strategies. The support of the bid distributiobn of the low types contains* 0. *If*

$$\frac{1 - e^{-\lambda_{0,l}}}{1 - e^{-\lambda_{1,l}}} < \frac{v(1)}{v(0)},$$

then the bid supports are non-overlapping intervals with a single point in common. If

$$\frac{1 - e^{-\lambda_{0,l}}}{1 - e^{-\lambda_{1,l}}} > \frac{v(1)}{v(0)},$$

then the bid supports intersect and 0 *is contained in the support of both types.*

3.4.2 Symmetric bidding equilibria in formal auctions

For completeness, we record here the following characterization of the unique symmetric equilibria in formal auctions from Chi, Murto, and Välimäki (2019).

Proposition 7 *In the formal first-price auction with two bidder types, there is a unique symmetric equilibrium. Given that there are $n \geq 2$ participants, low type bidders pool at bid*

$$\underline{p}(n) = E\left[v \left| \theta = l,\ N^l = n-1,\ N^h = 0 \right.\right] \text{ for } n \geq 2$$

and high types have an atomless bidding support $\left[\underline{p}(n), \overline{p}(n)\right]$ with some $\overline{p}(n) > \underline{p}(n)$. With $n = 1$, the only participant bids zero.

In the formal second-price auction, there is a unique symmetric equilibrium. In this equilibrium, low types pool at bid

$$\underline{p}(n) = E\left[v \left| \theta = l,\ N^l = n-1,\ N^h = 0 \right.\right]$$

and high types have an atomless bidding support $[p'(n), p''(n)]$, where

$$\begin{aligned} p'(n) &= \mathbb{E}\left[v \left| \theta = l,\ N^l = n-2,\ N^h = 1 \right.\right], \\ p''(n) &= \mathbb{E}\left[v \left| \theta = l,\ N^l \leq n-2,\ N^h \geq 1 \right.\right]. \end{aligned}$$

With $n = 1$, the only participant bids zero.

It is thus easy to compute the expected payoff at the entry stage if the bidding stage is in a formal auction. In the next two sections, we compare the overall revenues in the games where we account for both the bidding stage and the costly entry stage.

4 Endogenous Entry

In this section, we endogenize the entry decisions. By Proposition 5, we can pin down the expected equilibrium rents from all possible bids by considering models with binary types $\theta \in \{l, h\}$.We follow the same strategy as in Section 2 and analyze the properties of the interim entry equilibria through a comparison with the planner's solution.

4.1 Private and Social Incentives for Entry

We consider first a hypothetical game where the object is given for free to the entrant if there is a single entrant. With two or more entrants, the object is withdrawn from the market and the entrants just pay the entry cost. Denote by $V_\theta^0(\pi_h, \pi_l)$ the expected value of a bidder with signal θ at the auction stage when the symmetric entry profile is (π_h, π_l):

$$\begin{aligned} V_h^0(\pi_h, \pi_l) &= q_h e^{-\alpha\pi_h - (1-\alpha)\pi_l} v(1) + (1-q_h) e^{-\beta\pi_h - (1-\beta)\pi_l} v(0), \\ V_l^0(\pi_h, \pi_l) &= q_l e^{-\alpha\pi_h - (1-\alpha)\pi_l} v(1) + (1-q_l) e^{-\beta\pi_h - (1-\beta)\pi_l} v(0). \end{aligned}$$

Any (π_h, π_l) such that $V_h^0(\pi_h, \pi_l) = V_l^0(\pi_h, \pi_l) = c$ in the above equations guarantees that every potential entrant is indifferent between entering and staying out. By inspection, we see that this is the same condition as in (4), and hence private entry incentives coincide with social incentives. To understand why this is the case, note that in this hypothetical situation each entrant gets exactly her marginal contribution to the social welfare as her payoff. Since the value of the object is common to all the players, an entrant contributes to the total surplus if and only if she is the only entrant. Since the object is given for free in such a situation her private benefit equals her marginal contribution. Therefore, entry incentives coincide exactly with the social value of entry. As we show below, all the different auction mechanisms give (at least weakly) too high entry incentives for the high types. As in the two-bidder case, quantifying this excess incentive across different auction formats gives us our revenue comparisons.

In Figure 1, we illustrate the social optimum in the case with interior entry probabilities by drawing the planner's reaction curves

$$\begin{aligned}\pi_l^*(\pi_h) \quad &: \quad = \arg\max_{\pi_l \geq 0} W(\pi_l, \pi_h),\\ \pi_h^*(\pi_l) \quad &: \quad = \arg\max_{\pi_h \geq 0} W(\pi_l, \pi_h),\end{aligned}$$

in the $(\pi_h, \pi_l)-$plane. The social optimum is at the intersection of these curves. As shown above, these curves are also the indifference contours $V_h^0(\pi_h, \pi_l) = c$ and $V_l^0(\pi_h, \pi_l) = c$ for a potential entrant of type h and l, respectively, who gets her marginal contribution as expected payoff.

We will see that many of the auctions considered here give an additional reward to potential entrants on top of their social contribution. This distorts the entry rates from socially optimal levels. Since bidding zero is in the support of the low type bidders in the informal first-price auction, their equilibrium entry incentives coincide with the planner's incentives conditional on the entry rate of the high types. As a result, their equilibrium reaction curve coincides with the planner's reaction curve. For our revenue comparisons it is useful to analyze how distortions to the high types' entry rate change the total surplus. We denote by $W^*(\pi_h)$ the total surplus as a function of the high types' entry rate, when low types adjust entry optimally:

$$W^*(\pi_h) := \max_{\pi_l \geq 0} W(\pi_l, \pi_h).$$

The following Lemma shows that $W^*(\pi_h)$ is single peaked in π_h with its peak at π_h^{opt}. This is illustrated in Figure 1 by arrows along $\pi_l^*(\pi_h)$ that point towards increasing social surplus.

Lemma 8 *Let $\pi_h^{opt} > 0$ denote the socially optimal high type entry rate. We have*

$$\frac{dW^*(\pi_h)}{d\pi_h} \begin{cases} > 0 \text{ for } \pi_h < \pi_h^{opt} \\ = 0 \text{ for } \pi_h = \pi_h^{opt} \\ < 0 \text{ for } \pi_h > \pi_h^{opt} \end{cases}. \tag{5}$$

4.2 Interim Entry Equilibrium

Consider first the case where $c \geq \overline{c}$ and the planner's solution has $\pi_l^{opt} = 0$. Suppose that this is the case also in equilibrium. Since all the entering bidders have observed a high signal, it is easy to see that in all the auction formats that we consider, the expected payoff to the entering bidders is given by the probability that no other bidder entered times the expected value of the object conditional on that event. Since the updated belief of a high type bidder on $\{\omega = 1\}$ is given by $q_h = \frac{q\alpha}{q\alpha+(1-q)\beta}$, equating the expected benefit from entry to the cost of entry gives:

$$q_h e^{-\alpha\pi_h} v(1) + (1 - q_h) e^{-\beta\pi_h} v(0) = c.$$

This coincides with the planner's optimality condition. Hence the symmetric equilibrium entry profile in all of the auction formats that we consider coincides with the planner's optimal solution. The key reason for this is the lack of heterogeneity in the bidders' information.

In equilibrium the bidders are indifferent between entering and not entering. Hence their expected payoff must be at their outside option of 0. Since the auctions generate maximal social surplus (under the restriction to symmetric entry profiles) in this case and since bidders expected payoff is at zero, the seller collects the entire expected social surplus in expected revenues. Hence all these auction formats are also revenue maximizing within the class of symmetric mechanisms (where we require symmetry at the entry stage as well as at the bidding stage).

Proposition 9 *If $\overline{c} \leq c < \overline{\overline{c}}$, then only the high type enters and all the auction formats are efficient and hence revenue equivalent. If $c \geq \overline{\overline{c}}$, then there is no entry in the planner's solution nor in any auction format.*

We move next to the more interesting case where the planner's solution induces entry by both types. In this case we see immediately that the second-price auction leads to suboptimal entry decisions. This conclusion follows from a very simple argument showing that a bidder with a high signal earns a higher private benefit in the auction stage than their social contribution. In a model with common values, additional entry is socially valuable only if no other bidder participates in the auction. In a second price auction, the bidder with a high type gets the social benefit, but she also receives an extra information rent when bidding against bidders with low signals. This is an immediate consequence of the usual logic in models with affiliated values. As a result, entry models with a second price auction in the bidding stage feature excessive entry by the high types relative to the planner's solution.

Our main result is that the game with interim entry decisions followed by an informal first-price auction for the object has a unique symmetric equilibrium. This narrows down the set of possible equilibria in two ways. First, as we already showed in Proposition 4, there are no symmetric equilibria with pooling in the bidding stage. Given this, the atomless bidding equilibrium given in Proposition 6 is the unique candidate for the symmetric equilibrium in the bidding stage.

Second, we show the existence and uniqueness of equilibria for the entry stage. We denote by $V_{\theta}^{IF}(\pi_h, \pi_l)$ the expected payoff of a bidder with type θ at the bidding stage of the informal first-price auction for given entry rates π_h, π_l. We show that there is a unique pair $\left(\pi_h^{IF}, \pi_l^{IF}\right)$ such that $V_h^{IF}\left(\pi_h^{IF}, \pi_l^{IF}\right) = V_l^{IF}\left(\pi_h^{IF}, \pi_l^{IF}\right) = c$.

In addition to the uniqueness, the proposition also contains a qualitative statement about the equilibrium. A low type entrant gets a payoff in the bidding stage that is exactly her social contribution:

$$V_l^{IF}(\pi_h, \pi_l) = V_l^{0}(\pi_h, \pi_l),$$

which means that equilibrium entry point $\left(\pi_h^{IF}, \pi_l^{IF}\right)$ must be along the planner's reaction curve $\pi_l^{IF} = \pi_l^{*}\left(\pi_h^{IF}\right)$. A high type may get more, so that to dissipate excess rent of the high type, we must have $\pi_h^{IF} \geq \pi_l^{*}\left(\pi_l^{IF}\right)$. As a result, the equilibrium entry rate must be (weakly) too high for the high type and (weakly) too low for the low type:

Proposition 10 *The informal first-price auction with interim entry has a unique symmetric equilibrium with entry rates $\pi_h^{IF} \geq \pi_h^{opt}$ and $\pi_l^{IF} = \pi_l^{*}\left(\pi_h^{IF}\right) \leq \pi_l^{opt}$. All entering bidder types use atomless bidding strategies. Zero bids are always in the support of the low type bidders and the upper bound of the high type bidder support is given by*

$$b^{\max} := q_h v(1) + q_l v(0) - c.$$

5 Revenue comparisons

We compare here the expected revenue across auction formats. We assume throughout this section that $c < \overline{c}$. If $c \geq \overline{c}$, then all the auction formats are revenue equivalent as already stated in Proposition 9.

We showed in Proposition 5 above that in the case of informal first-price auction we can restrict our analysis to the two type model. The same is true for the formal auctions. In fact, we can show that the bidding equilibria of formal auctions are simpler than in the informal first-price auction in the sense that for all bid levels, either the lowest or the highest type has a strict advantage over any intermediate type. This implies that the intermediate types can never break even in equilibrium and so there cannot be any equilibria where intermediate types enter with positive rate.

Proposition 11 *Let* $\pi = (\pi_1, ..., \pi_M)$ *denote the vector of entry rates by different types in an equilibrium of the formal first-price auction or the formal second-price auction with interim entry. Then* $\pi_2 = ... = \pi_{M-1} = 0$*, i.e. only types* 1 *and* M *may enter with a strictly positive rate.*

Our first revenue comparison result shows that for some parameter values, the informal first price auction gives the entire (symmetric) social surplus to the seller in expected revenues. Hence the informal first price auction maximizes the seller's expected revenue in the class of all symmetric mechanisms. Since both types of formal auctions fall short of this revenue, we establish the strict superiority of the informal first price auction for this case.

Proposition 12 *If*

$$\frac{1-\beta}{1-\alpha} > \frac{v(1)}{v(0)}, \tag{6}$$

then there is a $c' < \bar{c}$ *such that for* $c \in (c', \bar{c})$*, entry is efficient in the informal first-price auction and the expected revenue is strictly higher than in any other symmetric auction format.*

When (6) does not hold, the revenue comparison is less straightforward as also the informal first-price auction induces too much entry by the high types. We show below that the basic insight of Proposition 12 continues to hold even if a more elaborate argument is needed.

As shown in Chi, Murto, and Välimäki (2019), first-price auction and second-price auction with two types generate the same expected revenue when the number of players is observed at the bidding stage. Together with Proposition 11 this implies that with interim entry, the two formal auctions are equivalent even when more than two types are allowed to enter. We show in the proof of Proposition 13 that if entry rates are not very high, then the insight that we obtained in the model with two potential bidders continues to hold: informal first-price auction leaves a lower rent to the bidders than the formal second-price auction (hence also the formal first-price auction). This in turn means that entry is less severely distorted in the informal first-price auction. Translated into the exogenous parameters of the model, this means that whenever entry cost is sufficiently high, informal first-price auction raises more revenue than the formal auctions:

Proposition 13 *There is some* $c'' < \bar{c}$ *such that for* $c \in (c'', \bar{c})$*, informal first-price auction generates a strictly higher expected revenue than formal auctions.*

6 Discussion

6.1 Affiliated Private Values and Independent Common Values

Even though we have restricted analysis to affiliated common values auctions, we should point out that a similar (and simpler) characterization for the informal first-price auction is available for the affiliated private values case. In some cases, the bid distributions of the two types overlap, and in these cases, equilibria are inefficient. If the bid supports do not overlap, then the informal auction is efficient and in this case, the expected payoff to all bidders coincides with their marginal contribution.

Revenue rankings in this case are not very surprising since formal second-price auctions result in the VCG -payoffs to all bidders (and with binary signals this is true for formal first-price auctions too).

If we assume common values but independent signals, then the bid distributions are non-overlapping and all the auction formats that we have discussed result in the same expected revenues.

6.2 Informal Second-Price Auctions

In our previous working paper, we also analyzed the informal second-price auction. The main difficulty for that analysis arises from the multiplicity of symmetric bidding equilibria. In contrast to informal first-price auctions, this multiplicity cannot be ruled out even in the case of interim entry decisions. However, it is clear that no matter how we select the equilibrium, the high type bidders always get a positive information rent on top of their social contribution, and hence entry rates are distorted. This means that the informal first-price auction dominates the informal second-price auction in expected revenue whenever the former has no distortions, that is, Proposition 12 holds as such.

We can further show the informal second-price auction has a bidder optimal equilibrium, i.e. there is a single Pareto efficient equilibrium for the bidders. If we select this equilibrium for the bidding stage, then we get unambiguous revenue ranking results between the two informal auction formats: the informal first-price auction raises a higher revenue than the informal second price auction.

6.3 Ex Ante Entry

In Appendix A, we discuss the results from the Poisson model with ex ante entry. As discussed above, the main complication for the analysis in this case is that also the informal first-price auction may have multiple symmetric equilibria. As long as we restrict to the comparison of the atomless symmetric equilibrium of the informal first-price auction with ex ante entry, our revenue ranking results are still valid: the informal first-price auction dominates the formal auctions for sufficiently high entry cost. For some parameter values, the atomless symmetric equilibrium of the informal first-price auction induces socially optimal entry and hence dominates all other auction formats with ex ante entry as well.

7 Appendix A: Ex Ante Entry Decisions with Two Types

In this appendix, we comment briefly on the alternative model specification where the probability of entry is determined prior to observing the type. In short, our results carry over with two caveats. The first one is that we cannot rule out alternative bidding equilibria with pooling, and hence the results are conditional on selecting the unique atomless bidding equilibrium. The second one is that the restriction to the case of only two types is not without loss. This is because if there are more types, ex-ante entry implies that they are all potentially present also in the bidding stage.

7.1 Two Bidders and Ex Ante Entry

The planner's problem is

$$\max_{0\leq\pi\leq 1}\left(1-(1-\pi)^2\right)\overline{v}-2\pi c,$$

where $\overline{v}=qv(1)+(1-q)\,v(0)$ is the ex ante expected value of the good. The interior solution to this problem is given by:

$$(1-\pi)\,\overline{v}=c \text{ or } \pi=\frac{\overline{v}-c}{\overline{v}}.$$

In other words, the planner equates the gain from adding a player to the cost. and notice that in this case, the solution is always interior if $\overline{v}>c$.

Recalling that our characterization result for the equilibrium bid distributions was derived under arbitrary (not necessarily equilibrium) entry rates, the same comments regarding the equilibrium revenue comparisons remain true in this case. The bidders can always secure their marginal contribution to the social surplus by making a zero bid at the bidding stage. Hence the equilibrium that minimizes the equilibrium payoffs in the bidding stage also results in the highest revenue.

We write the planner's expected gain from adding a player as:

$$(1-\pi)\,\overline{v}=\mathbb{E}_\theta(q_\theta\,(1-\pi)\,v\,(1)+(1-q_\theta)\,(1-\pi)\,v\,(0))$$

Since the expected payoff to the low type bidder is $q_l\,(1-\pi)\,v\,(1)+(1-q_l)\,(1-\pi)\,v\,(0)$ in any of the auction formats that we consider, we see that the size of the distortion at the entry stage depends only on the equilibrium payoff to the high type bidder at entry rate π. Since formal auctions induce a higher payoff to high type bidders, we see that the informal first-price auction dominates formal auctions in terms of expected revenue.

7.2 Poisson Entry: Planner's problem

When entry decisions are taken at the ex ante stage, the equilibrium determination of entry rates is straightforward. Since all the players are ex ante symmetric, only a single entry rate π needs to be determined. As in the main text, we start with the socially optimal choice of π. Since the expected number of entrants is equal to the Poisson parameter π, the planner's problem is then to

$$\max_{\pi \geq 0} \overline{v}\left(1 - e^{-\pi}\right) - \pi c,$$

where $\overline{v} = qv(1) + (1-q)v(0)$. Note that the marginal benefit from increasing the entry intensity is the probability that there are no entrants times the value of the object.

This problem has a strictly concave objective function and since we are assuming $v(0) > c$, it has an interior solution

$$\pi^* = \ln\left(\frac{\overline{v}}{c}\right).$$

It is also clear that the entry stage for any of our four auction formats will have a unique interior entry rate that balances the cost and benefit of entry and keeps the potential entrants indifferent between entering and not entering. This means that we can rank the expected revenue of our auction formats by computing the social surplus induced equilibrium entry rates as before. By the concavity we know that if $\pi^{FA} > \pi^{IF} \geq \pi^*$, then the informal first-price auction dominates the formal auctions in terms of expected revenue.

7.3 Bidding Equilibrium and Ex-Ante Entry

Since entry decisions are taken at a stage where all potential entrants are symmetrically informed about the value of the product, the realized number of entrants conveys no information. This distinguishes the current ex ante entry model from the interim entry model of the main text. Nevertheless if the two types of bidders use different strategies at the bidding stage, then winning the auction conveys information about the realized types of the other bidders and hence about the true value of the object.

In the proof of Proposition 4, we ruled out pooling equilibria by using the fact that both types of bidders are indifferent between entering and not entering. Unfortunately this condition is not available in the case of ex ante entry and we have not been able to rule out pooling equilibria generally. Nevertheless the steps in the proof of Proposition 6 demonstrating the uniqueness in symmetric atomless equilibria applies equally well for the special case where $\pi_h = \pi_l$.

Concentrating on this unique atomless bidding equilibrium of the informal first-price auction gives us revenue comparisons very similar to the case of interim entry. It is easy to show that there is an equilibrium (b, π) to the full model with ex-ante entry, where b defines an atomless bidding distribution $F_\theta(\cdot)$ for $\theta = l, h$, and the entry rate π keeps all the entrants indifferent between entering and not. Proposition 6 says that 0 is in the support of $F_l(\cdot)$ and implies that the low type bidders' expected payoff coincides with the value of the object conditional on being the only participating bidder in the auction. Further, it remains true that 0 is also in the support of $F_h(\cdot)$ whenever

$$\frac{1-\beta}{1-\alpha} > \frac{v(1)}{v(0)}$$

and c is sufficiently high. It follows that under this condition, informal first-price auction gives the maximal surplus to the seller and hence the conclusion of Proposition 12 holds for ex-ante entry as well. We can also show that the proof of Proposition 13 carries over to the case of ex-ante entry, and hence even when entry is distorted in informal first-price auction, that format gives a higher revenue than the formal auctions if c is sufficiently high.

7.4 Appendix B: Proofs

Proof of Proposition 1. We can rule out atoms and gaps in the union of the bidding distributions by standard arguments. If there was an atom, then a slight deviation up from the atom would increase the winning probability by a discrete amount and hence improve expected payoff (unless the atom is so high that expected payoff is negative, in which case deviation to zero would naturally be profitable). Similarly, we can rule out gaps in the union of the supports as follows. Suppose $(p_1, p_2) \notin \text{supp}F_l(\cdot) \cup \text{supp}F_h(\cdot)$ and $p_2 \in \text{supp}F_l(\cdot) \cup \text{supp}F_h(\cdot)$. Then the type bidding p_2 benefits strictly by lowering bid to p_1. It follows than in equilibrium both bidders use atomless mixed strategies with distributions that satisfy $\text{supp}F_l(\cdot) \cup \text{supp}F_h(\cdot) = [0, p^{\max}]$. Note that we have not ruled out gaps in the supports of individual types, $\text{supp}F_l$ or $\text{supp}F_h$.

We proceed by analyzing the properties of equilibrium payoff functions

$$U_\theta(p) = q_\theta R_1(p) + (1-q_\theta) R_0(p),$$

where $R_\omega(p)$ is the expected payoff from bidding p conditional on state ω:

$$\begin{aligned} R_1(p) &= (1 - \alpha\pi_h(1-F_h(p)) - (1-\alpha)\pi_l(1-F_l(p)))(v(1)-p), \\ R_0(p) &= (1 - \beta\pi_h(1-F_h(p)) - (1-\beta)\pi_l(1-F_l(p)))(v(0)-p). \end{aligned}$$

We say that type θ is active at p when $p \in \text{supp}F_\theta$. In equilibrium we must have

$$U_\theta(p) \begin{cases} = V_\theta \text{ for } p \in \text{supp}F_\theta \\ \leq V_\theta \text{ for } p \notin \text{supp}F_\theta \end{cases} \tag{7}$$

where V_θ is the equilibrium payoff of type θ. A player can always guarantee payoff $U_\theta(0) > 0$ by bidding zero (since $\pi_\theta < 1$ for $\theta \in \{l, h\}$), which gives a lower bound for equilibrium payoffs:

$$\begin{aligned} V_\theta \geq\ & U_\theta(0) = q_\theta(1 - \alpha\pi_h - (1-\alpha)\pi_l)v(1) \\ & + (1-q_\theta)(1 - \beta\pi_h - (1-\beta)\pi_l)v(0). \end{aligned}$$

As a first step, we determine the curvature of $U_\theta(p)$ for regions where type θ is not active. In particular, we show that $U_h(p)$ is convex and $U_l(p)$ is concave in any inactive region.

Differentiating $U_\theta(p)$ once and twice gives

$$\begin{aligned} U'_\theta(p) &= q_\theta R'_1(p) + (1-q_\theta) R'_0(p) = \\ & q_\theta[(\alpha\pi_h F'_h(p) + (1-\alpha)\pi_l F'_l(p))(v(1)-p) - (1-\alpha\pi_h(1-F_h(p)) - (1-\alpha)\pi_l(1-F_l(p)) \\ & +(1-q_\theta)[(\beta\pi_h F'_h(p) + (1-\beta)\pi_l F'_l(p))(v(0)-p) - (1-\beta\pi_h(1-F_h(p)) - (1-\beta)\pi_l(1- \end{aligned}$$

and

$$\begin{aligned} U''_\theta(p) &= q_\theta R''_1(p) + (1-q_\theta) R''_0(p) = \\ &= q_\theta[\alpha\pi_h(F''_h(p)(v(1)-p) - 2F'_h(p)) + (1-\alpha)\pi_l(F''_l(p)(v(1)-p) - 2F'_l(p))] \\ & +(1-q_\theta)[\beta\pi_h(F''_h(p)(v(0)-p) - 2F'_h(p)) + (1-\beta)\pi_l(F''_l(p)(v(0)-p) - 2F'_l(p))] \end{aligned} \quad (9)$$

Suppose first that type h is inactive for some p, $0 \le p \le p^{\max}$. Then $F'_h(p) = F''_h(p) = 0$. Since the union of the two bid supports is connected, l must be active at p and we have $F'_l(p) > 0$ and $U''_l(p) = 0$. This implies that one of the terms $R''_1(p)$ and $R''_0(p)$ must be positive, and the other must be negative. Since $1-\beta > 1-\alpha$ and $v(1) > v(0)$, it follows from (9) that $R''_1(p) > R''_0(p)$ in this case. Since $q_h > q_l$, this implies that $U''_h(p) > 0$. Going through the same steps we see that whenever $F'_l(p) = F''_l(p) = 0$ and $F'_h(p) > 0$, we have $U''_l(p) < 0$. To summarize: if there is an interval where type h is inactive, the value of that type is convex: $U''_h(p) > 0$ for $p \notin \mathrm{supp} F_h$. If there is an interval where type l is inactive, the value of that type is concave: $U''_l(p) < 0$ for $p \notin \mathrm{supp} F_l$. This has an immediate implication for the types of equilibria that are possible. The strict concavity of $U_l(p)$ on all intervals where l is inactive rules out gaps in $\mathrm{supp} F_l$ since the expected payoffs at the endpoints of such a gap are equal in any equilibrium. In contrast, there may be a gap in $\mathrm{supp} F_h$.

We show next that $0 \in \mathrm{supp} F_l$. Suppose to the contrary that $0 \notin \mathrm{supp} F_l$. Let $\widetilde{p} := \min \mathrm{supp} F_l > 0$ denote the lowest bid by the low type. Then for $p \in [0, \widetilde{p}]$ we must have $U_h(p) = V_h$ and $U'_h(p) = 0$. Noting that $\alpha > \beta$ and $F'_l(p) = 0$ for $p \in [0, \widetilde{p}]$, we see from (8) that $R'_1(p) > R'_0(p)$ for p small enough. Since $q_h > q_l$ and $U'_h(p) = 0$, this implies that $U'_l(p) < 0$ for p small enough. We have also shown above that $U''_l(p) < 0$ when l is inactive, in particular for $p \in [0, \widetilde{p}]$, and therefore $U_l(\widetilde{p}) < U_l(0)$. This means that the low type gets a strict benefit by deviating to zero, contradicting $\widetilde{p} := \min \mathrm{supp} F_l$. We can conclude that in equilibrium $\mathrm{supp} F_l$ is an interval containing zero.

The remaining task is to determine the shape of the support of the high types' bid distribution in equilibrium. We first attempt to construct an equilibrium where $\mathrm{supp} F_l(\cdot) = [0, p']$ and $\mathrm{supp} F_h(\cdot) = [p', p^{\max}]$ for some $p' > 0$. By standard analysis, solving (8) for $F'_l(p)$ while keeping $F_h(p) = F'_h(p) = 0$, we can find a $p' > 0$ and $F'_l(p) > 0$ for all $p \in [0, p']$ in such a way that $U'_l(p) = 0$ for $p \in [0, p']$

and where $F_l(p') = 1$. Similarly, we can find a $p^{\max} > p'$ and $F_h'(p)$ for $p \in [0, p^{\max}]$ such that $U_h'(p) = 0$ for $p \in [0, p^{\max}]$.

In this construction it is evident that $U_l(p) < U(0)$ for all $p > p'$ and hence the low type has no profitable deviation. Since $U_h(p)$ is convex in $[0, p']$, we have to check that the high type does not want to deviate to zero, i.e. $U_h(p') \geq U_h(0)$. To do this, we can pin down p' by requiring the low type to be indifferent between 0 and p':

$$\begin{aligned} & q_l(1 - \alpha\pi_h - (1-\alpha)\pi_l)v(1) + (1-q_l)(1 - \beta\pi_h - (1-\beta)\pi_l)v(0) \\ = \; & q_l(1 - \alpha\pi_h)(v(1) - p') + (1-q_l)(1 - \beta\pi_h)(v(0) - p'). \end{aligned}$$

Solving this for p' gives

$$p' = \pi_l \frac{q_l(1-\alpha)v(1) + (1-q_l)(1-\beta)v(0)}{q_l(1-\alpha\pi_h) + (1-q_l)(1-\beta\pi_h)}. \tag{10}$$

The values of the high type for $p = 0$ and $p = p'$ are then, respectively:

$$U_h(0) = q_h(1 - \alpha\pi_h - (1-\alpha)\pi_l)v(1) + (1-q_h)(1 - \beta\pi_h - (1-\beta)\pi_l)v(0),$$

$$U_h(p') = q_h(1 - \alpha\pi_h)(v(1) - p') + (1-q_h)(1 - \beta\pi_h)(v(0) - p').$$

Substituting in p' from (10) and using straightforward algebra, we then see that $U_h(p') \geq U_h(0)$ whenever:

$$\frac{(1-\alpha)v(1)}{1-\alpha\pi_h} \geq \frac{(1-\beta)v(0)}{1-\beta\pi_h}, \tag{11}$$

and $U_h(p)$ is convex in $[0, p']$, we have $U_h(p) \leq U_h(p')$ for all $p \in [0, p']$. If (11) holds, our constructed strategy profile is then an equilibrium with $\text{supp}F_l = [0, p']$ and $\text{supp}F_h = [p', p^{\max}]$. Using the properties (7) of U_θ that must hold in equilibrium it is straight-forward to also show that there cannot be any other equilibrium.

If (11) does not hold, then $U_h(0) > U_h(p')$ in the strategy profile that we constructed above, and the high types have a profitable deviation. In that case, equilibrium must satisfy $0 \in \text{supp}F_h$. To see this, note that by convexity of $U_h(p)$ in the inaction region of h, we have necessarily $U_h(0) > U_h(\widetilde{p})$ for any potential $\widetilde{p} := \min\text{supp}F_h \in (0, p']$ (since there cannot be a gap in the union of supports, obviously we cannot have $\widetilde{p} > p'$ either). When (11) does not hold, we can construct a unique equilibrium where either 1) $\text{supp}F_l = [0, p']$ and $\text{supp}F_h = [0, p^{\max}]$, $0 < p' < p^{\max}$, or 2) $\text{supp}F_l = [0, p']$ and $\text{supp}F_h = [0, p''] \cup [p', p^{\max}]$, $0 < p'' < p' < p^{\max}$. The construction of such an equilibrium follows exactly the lines of the proof of Proposition 6 and we skip the details here. ■

Proof of Proposition 2. We contrast the expected revenue in the informal first-price auction to the formal second-price auction. Since the expected revenue is the same in both formal auctions, the proof extends to the formal first-price auction as well.

By Proposition 1, there are two cases to consider. If $0 \in \text{supp}F_l(\cdot) \cap \text{supp}F_h(\cdot)$ in the equilibrium of the informal first-price auction, both type of bidders get expected payoff $U_\theta(0)$, i.e. probability that the other bidder does not enter time the expected value of the object conditional on that. In the formal second price auction, a high type bidder gets a strictly higher expected payoff, because even when both bidders enter, the high type pays less than the expected value if the other bidder has type $\theta = l$. It follows that the expected rent to the high type is higher in the formal second-price auction than the informal first-price auction and hence the expected revenue is higher in the informal first-price auction.

Consider next the case where $\text{supp}F_l(\cdot) = [0, p']$ and $\text{supp}F_h(\cdot) = [p', p^{\max}]$ for some $p' > 0$ in the informal first-price auction. We want to contrast the equilibrium payoff of the high type bidder in the informal first-price auction to the formal second-price auction. We introduce the notation $N^\theta \in \{0, 1\}$ for the random number of participating other bidders of type θ.

Notice first that by bidding 0 in the informal first-price auction, the low type bidder wins if and only if no other bidders enter. By bidding p', she wins if and only if no high type bidder enters. Since both of these bids are in her bid support, they must yield the same expected payoff:

$$
\begin{aligned}
U_l(0) &= \Pr\left(N^l = 0 \left| \widetilde{\theta} = l, N^h = 0\right.\right) \mathbb{E}\left(v \left| \widetilde{\theta} = l, N^h = 0, N^l = 0\right.\right) - 0 = \\
U_l(p') &= \Pr\left(N^l = 0 \left| \widetilde{\theta} = l, N^h = 0\right.\right) \mathbb{E}\left(v \left| v = l, N^h = 0, N^l = 0\right.\right) \\
&\quad + \Pr\left(N^l = 1 \left| \widetilde{\theta} = l, N^h = 0\right.\right) \mathbb{E}\left(v \left| \widetilde{\theta} = l, N^h = 0, N^l > 0\right.\right) - p'
\end{aligned}
$$

so that

$$
p' = \Pr\left(N^l = 1 \left| \widetilde{\theta} = l, N^h = 0\right.\right) \mathbb{E}\left(v \left| \widetilde{\theta} = l, N^h = 0, N^l = 1\right.\right).
$$

The bid p' is also in the support of the high type bidder.

Consider then the formal second-price auction, where type θ bids b_θ in equilibrium. Observe that by deviating to some bid between b_l and b_h, the high type bidder does not change her equilibrium payoff (as long as ε is small enough). At this deviating bid, the allocation of the deviating bidder is exactly the same as allocation from the equilibrium bid of p' in the informal first-price auction. To compare the expected payoff to the high type bidder across the two auction formats, we then only need to compare the expected payment in these formats.

The expected payment of the deviating high type bidder in the formal second-price auction is

$$
\begin{aligned}
\mathbb{E}(p) &= \Pr\left(N^l = 0 \left| \widetilde{\theta} = h, N^h = 0\right.\right) \cdot 0 + \Pr\left(N^l = 1 \left| \widetilde{\theta} = h, N^h = 0\right.\right) b_l \\
&= \Pr\left(N^l = 1 \left| \widetilde{\theta} = h, N^h = 0\right.\right) \mathbb{E}\left(v \left| \widetilde{\theta} = l, N^h = 0, N^l = 1\right.\right),
\end{aligned}
$$

where we have used the fact that bid of a low type in the formal second-price auction is

$$
b_l = \mathbb{E}\left(v \left| \widetilde{\theta} = l, N^h = 0, N^l = 1\right.\right).
$$

Observe that

$$\Pr\left(N^l = 1 \middle| \widetilde{\theta} = h, N^h = 0\right) < \Pr\left(N^l = 1 \middle| \widetilde{\theta} = l, N^h = 0\right)$$

because the bidder with signal $\widetilde{\theta} = h$ considers state $\widetilde{\omega} = 1$ more likely than the bidder with signal $\widetilde{\theta} = l$ and $(1 - \alpha) < (1 - \beta)$. By comparing the expressions above, we then note that $\mathbb{E}(p) < p'$. Since the expected payment of a high type is lower in the formal second-price auction, it follows that expected rents to the bidders are lower in the informal first-price auction, or equivalently, the informal first-price auction generates a higher expected revenue than the formal second-price auction. ■

Proof of Proposition 3. Let $\left(\pi_h^{IF}, \pi_l^{IF}\right)$ and $\left(\pi_h^F, \pi_l^F\right)$ denote the equilibrium entry probabilities in informal first-price auction and formal second-price auction, respectively. Let $V_\theta^{IF}(\pi_h, \pi_l)$ and $V_\theta^F(\pi_h, \pi_l)$ denote the equilibrium value of type θ in the informal first-price and formal second-price auction, respectively, when entry rates are exogenously given by (π_h, π_l). Fix the entry probabilities to those given by the entry equilibrium of the informal first-price auction, $\left(\pi_h^{IF}, \pi_l^{IF}\right)$, but change the auction format to the formal second-price auction. By Proposition 2, the rent of the higher bidder exceeds that in the informal first-price auction, and hence $V_h^F\left(\pi_h^{IF}, \pi_l^{IF}\right) > c$. If $\pi_h^{IF} < 1$, then in the entry equilibrium of the formal second-price auction we must have $\pi_h^F > \pi_h^{IF}$. Since $\pi_l^F = \pi_l^*\left(\pi_h^F\right)$, the result follows. If $\pi_h^{IF} = 1$, then the equilibrium entry rates are the same across the two auction formats (and therefore the realized social surplus is also the same across the formats), but the rent going to the high type bidders is higher in the informal second-price auction. ■

Proof of Proposition 5. If $\left(\overline{b}, (\overline{\pi}_1, \overline{\pi}_M)\right)$ is an equilibrium of the reduced model (i.e. the model with $\theta \in \{1, M\}$), then types $\theta = 1$ and $\theta = M$ get the same payoff in the full model under strategy profile $\left(\overline{b}, (\overline{\pi}_1, 0, ..., 0, \overline{\pi}_M)\right)$ and hence their behavior remains consistent with equilibrium. We have to prove that no other type $m \in \{2, ..., M-1\}$ has a strict incentive to enter against $\left(\overline{b}, (\overline{\pi}_1, 0, ..., 0, \overline{\pi}_M)\right)$. Denote by $\overline{R}_0(p)$ and $\overline{R}_1(p)$ the expected payoff of a bidder who bids p against $\left(\overline{b}, (\overline{\pi}_1, 0, ..., 0, \overline{\pi}_M)\right)$, conditional on state $\omega = 0$ and $\omega = 1$, respectively. Suppose that there is a type $m \in \{2, ..., M-1\}$ and a bid p_m such that m gets strictly more than c when the others follow $\left(\overline{b}, (\overline{\pi}_1, 0, ..., 0, \overline{\pi}_M)\right)$. This means that $q_m \overline{R}_1(p_m) + (1 - q_m)\overline{R}_0(p_m) > c$. But since $q_0 < q_m < q_M$, then either $q_1 \overline{R}_1(p_m) + (1 - q_0)\overline{R}_0(p_m) > c$ or $q_M \overline{R}_1(p_m) + (1 - q_M)\overline{R}_0(p_m) > c$, implying that either type $\theta = 0$ or $\theta = M$ would get a payoff strictly greater than c by following bidding strategy p_m. This contradicts the fact that $\left(\overline{b}, (\overline{\pi}_1, \overline{\pi}_M)\right)$ is an equilibrium of the reduced model.

To prove the second part of the proposition, suppose that there is an equilibrium (b, π) in the full model such that $\pi_m > 0$ for some $m \in \{2, ..., M-1\}$. Let $R_\omega(p)$ denote the expected payoff of bidding p conditional on state ω:

$$R_1(p) = e^{-\sum_{m=1}^{M} \alpha_{1,m}\pi_m(1-F_m(p))} (v(1) - p),$$
$$R_0(p) = e^{-\sum_{m=1}^{M} \alpha_{0,m}\pi_m(1-F_m(p))} (v(0) - p).$$

Since (b, π) is an equilibrium, we must have for every $p \geq 0$,

$$q_m R_1(p) + (1 - q_m) R_0(p) \leq c, \text{ for } m = 1, ..., M$$

and for any $p \in \text{supp} F_m$, we have

$$q_m R_1(p) + (1 - q_m) R_0(p) = c.$$

These equations imply that if $p \in \text{supp} F_m$, $m \in \{2, ..., M-1\}$, then

$$R_1(p) = R_0(p) = c,$$

and hence any type will be indifferent between entering and bidding p and staying out. It follows that if there is a strategy profile $(\overline{b}, (\overline{\pi}_1, \overline{\pi}_M))$ in the reduced model that induces the same state-by-state payoffs $R_1(p)$ and $R_0(p)$ than (b, π), then $(\overline{b}, (\overline{\pi}_1, \overline{\pi}_M))$ is an equilibrium of the reduced model. It remains to show that we can construct such a strategy profile.

Since $\frac{\alpha_{1,1}}{\alpha_{0,1}} < ... < \frac{\alpha_{1,M}}{\alpha_{0,M}}$, we can always choose $\widehat{\pi}_1 > 0$ and $\widehat{\pi}_M > 0$ such that

$$\alpha_{1,1}\widehat{\pi}_1 + \alpha_{1,M}\widehat{\pi}_M = \sum_{m=1}^{M} \alpha_{1,m}\pi_m,$$
$$\alpha_{0,1}\widehat{\pi}_1 + \alpha_{0,M}\widehat{\pi}_M = \sum_{m=1}^{M} \alpha_{0,m}\pi_m.$$

Similarly, we choose c.d.f.:s $\widehat{F}_1(p)$ and $\widehat{F}_M(p)$ such that for all $p \geq 0$, we have

$$\alpha_{1,1}\widehat{\pi}_1\left(1 - \widehat{F}_1(p)\right) + \alpha_{1,M}\widehat{\pi}_M\left(1 - \widehat{F}_M(p)\right) = \sum_{m=1}^{M} \alpha_{1,m}\pi_m(1 - F_m(p)),$$
$$\alpha_{0,1}\widehat{\pi}_1\left(1 - \widehat{F}_1(p)\right) + \alpha_{0,M}\widehat{\pi}_M\left(1 - \widehat{F}_M(p)\right) = \sum_{m=1}^{M} \alpha_{0,m}\pi_m(1 - F_m(p)).$$

We now have a strategy profile of the reduced model, $\left(\left\{\widehat{F}_1(\cdot), \widehat{F}_M(\cdot)\right\}, \{\widehat{\pi}_1, \widehat{\pi}_M\}\right)$, with the same state-by-state payoffs as the original equilibrium of the full model:

$$\widehat{R}_1(p) := e^{-\alpha_{1,1}\widehat{\pi}_1\left(1-\widehat{F}_1(p)\right)-\alpha_{1,M}\widehat{\pi}_M\left(1-\widehat{F}_M(p)\right)} (v(1) - p) = R_1(p),$$
$$\widehat{R}_0(p) := e^{-\alpha_{0,1}\widehat{\pi}_1\left(1-\widehat{F}_1(p)\right)-\alpha_{0,M}\widehat{\pi}_M\left(1-\widehat{F}_M(p)\right)} (v(0) - p) = R_0(p).$$

■

Proof of Proposition 6. We prove the proposition by constructing the equilibrium bidding functions by requiring indifference over intervals of bids. Since the equations determining this indifference have a unique solution, we get the uniqueness of atomless bidding equilibria as a by-product of this procedure.

To begin, we specify the range of bids where both types can potentially be indifferent simultaneously. Let us denote by $U_\theta(p)$ the payoff of type θ who bids p, when bidding distributions are given by $F_\theta(p)$:

$$\begin{aligned} U_\theta(p) &= q_\theta e^{-\lambda_{1,h}(1-F_h(p))-\lambda_{1,l}(1-F_l(p))}(v(1)-p) \\ &\quad +(1-q_\theta)\, e^{-\lambda_{0,h}(1-F_h(p))-\lambda_{0,l}(1-F_l(p))}(v(0)-p). \end{aligned}$$

Differentiating with respect to p, we have:

$$U'_\theta(p) = q_\theta e^{-\lambda_{1,h}(1-F_h(p))-\lambda_{1,l}(1-F_l(p))}\left[\left(F'_h(p)\lambda_{1,h}+F'_l(p)\lambda_{1,l}\right)(v(1)-p)-1\right]$$

$$+(1-q_\theta)\, e^{-\lambda_{0,h}(1-F_h(p))-\lambda_{0,l}(1-F_l(p))}\left[\left(F'_h(p)\lambda_{0,h}+F'_l(p)\lambda_{0,l}\right)(v(0)-p)-1\right] \tag{12}$$

In equilibrium, $F'_\theta(p)>0$ requires $U'_\theta(p)=0$ to maintain indifference within bidding support.

To analyze when this can hold, we denote the two terms in square brackets by $B(1)$ and $B(0)$:

$$\begin{aligned} B(1) &= \left(F'_h(p)\lambda_{1,h}+F'_l(p)\lambda_{1,l}\right)(v(1)-p)-1, \\ B(0) &= \left(F'_h(p)\lambda_{0,h}+F'_l(p)\lambda_{0,l}\right)(v(0)-p)-1. \end{aligned}$$

These terms are weighted in (12) by positive terms that depend on θ only through q_θ. Since $q_h > q_l$, we note that $U'_h(p)$ puts more weight on term $B(1)$ than $B(0)$, relative to $U'_l(p)$.

It is immediate that for $U'_h(p)$ and $U'_l(p)$ to be zero simultaneously, it must be that $B(1) = B(0) = 0$. Since $\lambda_{1,h} > \lambda_{0,h}$ and $\lambda_{0,l} > \lambda_{1,l}$, this is possible only if

$$\lambda_{0,l}(v(0)-p) > \lambda_{1,l}(v(0)-p).$$

This can hold only for low values of p. Let us define $\widetilde{p}$ as the cutoff value such that the above inequality holds for $p < \widetilde{p}$:

$$\widetilde{p} = \max\left(0, \frac{v(0)\lambda_{0,l}-v(1)\lambda_{1,l}}{\lambda_{0,l}-\lambda_{1,l}}\right).$$

(Note that we define $\widetilde{p}=0$ if indifference is never possible). We summarize the implications of this reasoning in the following Lemma. Part 1 says that the overlap of bidding supports is possible only for $p<\widetilde{p}$. Part 2 says that in a range where only low type randomizes, value of high type is U-shaped with minimum at $p=\widetilde{p}$. Part 3 says that in a range where only high type randomizes, value of low type is decreasing and hence the low type prefers lower bids.

Lemma 14 *Let $\lambda_{1,h} > \lambda_{0,h}$ and $\lambda_{0,l} > \lambda_{1,l}$ be given, and let $F_\theta(p)$, $\theta = h, l$, be an atomless equilibrium bidding distribution. Then:*

1. If $F'_\theta(p) > 0$ for $\theta = h, l$, then $p < \widetilde{p}$.
2. If $F'_l(p) > 0$ and $F'_h(p) = 0$, then
$$U'_h(p) \begin{cases} < 0 \text{ for } p < \widetilde{p} \\ > 0 \text{ for } p > \widetilde{p} \end{cases}.$$
3. If $F'_h(p) > 0$ and $F'_l(p) = 0$, then $U'_l(p) < 0$.

Proof. Part 1: $F'_\theta(p) > 0$ for $\theta = h, l$ requires that $B(1) = B(0) = 0$, which is only possible if $p < \widetilde{p}$. Part 2: If $F'_l(p) > 0$, then $U'_l(p) = 0$. If $F'_h(p) = 0$, then $U'_l(p) = 0$ implies that $B(1) < 0 < B(0)$ for $p < \widetilde{p}$, and $B(0) < 0 < B(1)$ for $p > \widetilde{p}$. Since $q_h > q_l$, the result follows. Part 3: If $F'_h(p) > 0$, then $U'_h(p) = 0$. If $F'_l(p) = 0$, then $U'_h(p) = 0$ implies that $B(0) < 0 < B(1)$. Since $q_l < q_h$, the result follows. ■

With this preliminary result in place, we can construct the equilibrium in the two cases separately. Assume first that
$$\left(1 - e^{-\lambda_{1,l}}\right) v(1) > \left(1 - e^{-\lambda_{0,l}}\right) v(0).$$

Starting from $p = 0$, assume that only low types bid for low p and define the low type bidding distribution $F_l(p)$ over some interval $[0, p']$ in such a way that a low type bidder is indifferent throughout, and $F_l(p') = 1$. In particular, a low type bidder must be indifferent between bidding 0 and p', which gives the following condition:

$$\begin{aligned} & q_l e^{-\lambda_{1,h} - \lambda_{1,l}} v(1) + (1 - q_l) e^{-\lambda_{0,h} - \lambda_{0,l}} v(0) \\ = \; & q_l e^{-\lambda_{1,h}} \left(v(1) - p'\right) + (1 - q_l) e^{-\lambda_{0,h}} \left(v(0) - p'\right), \end{aligned}$$

which can be rewritten as

$$q_l e^{-\lambda_{1,h}} \left[\left(1 - e^{-\lambda_{1,l}}\right) v(1) - p'\right] + (1 - q_l) e^{-\lambda_{0,h}} \left[\left(1 - e^{-\lambda_{0,l}}\right) v(0) - p'\right] = 0.$$

For this to hold, one of the terms in square-brackets must be positive and the other one must be negative. Since we assume $\left(1 - e^{-\lambda_{1,l}}\right) v(1) > \left(1 - e^{-\lambda_{0,l}}\right) v(0)$, it must be the term $\left[\left(1 - e^{-\lambda_{1,l}}\right) v(1) - p'\right]$ that is positive. But then, since $q_h > q_l$, this implies that

$$q_h e^{-\lambda_{1,h}} \left[\left(1 - e^{-\lambda_{1,l}}\right) v(1) - p'\right] + (1 - q_h) e^{-\lambda_{0,h}} \left[\left(1 - e^{-\lambda_{0,l}}\right) v(0) - p'\right] > 0$$

or

$$\begin{aligned}
& q_h e^{-\lambda_{1,h}-\lambda_{1,l}} v(1) + (1-q_h) e^{-\lambda_{0,h}-\lambda_{0,l}} v(0) \\
< \; & q_h e^{-\lambda_{1,h}} (v(1) - p') + (1-q_h) e^{-\lambda_{0,h}} (v(0) - p'),
\end{aligned}$$

so that high types prefer strictly the bid of p' to the bid 0. Combining this with part 2 of Lemma 14, we note that

$$U_h(p') > U_h(p) \text{ for all } p \in [0, p')$$

and so the high type does not have an incentive to deviate to any $p < p'$.

To finish the construction of the equilibrium, we define $p^{\max}$ so that the high type is indifferent between winning for sure by bidding $p^{\max}$ and winning if and only there are no other high types by bidding p':

$$\begin{aligned}
& q_h (v(1) - p^{\max}) + (1-q_h)(v(0) - p^{\max}) \\
= \; & q_h e^{-\lambda_{1,h}} (v(1) - p') + (1-q_h) e^{-\lambda_{0,h}} (v(0) - p').
\end{aligned}$$

It is then clear that there is a unique distribution $F_h(p)$ such that $F_h(p') = 0$, $F_h(p^{\max}) = 1$, and for which high type is indifferent between any $p \in [p', p^{\max}]$:

$$U_h(p) = U_h(p') \text{ for all } p \in (p', p^{\max}].$$

We have hence constructed an equilibrium where low type bidding support is $[0, p']$ and high type bidding support is $[p', p^{\max}]$.

Assume next that

$$\left(1 - e^{-\lambda_{1,l}}\right) v(1) < \left(1 - e^{-\lambda_{0,l}}\right) v(0).$$

If we now try to construct an equilibrium as above, high type bidders have an incentive to deviate and bid zero. In particular, take any $p' > 0$ and assume that only low types bid on interval $[0, p']$ and $U_l(p) = U_l(0)$ for all $p \leq p'$. Then by part 1 of Lemma 14 we have $U_h(p) < U_h(0)$ for all $p \leq p'$, and high types will deviate to zero. It follows that in any equilibrium with atomless distributions, 0 must be contained in the bidding distributions of both types. We can now construct the equilibrium bidding distributions as follows. First, we define p'' such that a high type bidder is indifferent between winning for sure by bidding p'' and winning if and only if there are no other bidders by bidding 0:

$$\begin{aligned}
& q_h (v(1) - p'') + (1-q_h)(v(0) - p'') \\
= \; & q_h e^{-\lambda_{1,h}} e^{-\lambda_{1,l}} v(1) + (1-q_h) e^{-\lambda_{0,h}} e^{-\lambda_{0,l}} v(0),
\end{aligned}$$

so that

$$p'' = q_h \left(1 - e^{-\lambda_{1,h} - \lambda_{1,l}}\right) v(1) + (1 - q_h) \left(1 - e^{-\lambda_{0,h} - \lambda_{0,l}}\right) v(0).$$

Then we can proceed downwards from p'' by defining $F_h(p)$ in such a way that

$$U_h(p) = U_h(p'') \text{ for } p < p'',$$

where $U_h(p)$ is given by (**??**) with $F_l(p) = 1$. At the same time, by part 3 of Lemma 14, we have $U_l'(p) < 0$ over this range, and at some point p will reach a point $\overrightarrow{p}$ where low types want to become active. This point is pinned down by the condition that low type is indifferent between bidding $\overrightarrow{p}$ and zero:

$$\overrightarrow{p} = \left\{p : U_l(p) = q_l e^{-\lambda_{1,h}} e^{-\lambda_{1,l}} v(1) + (1 - q_l) e^{-\lambda_{0,h}} e^{-\lambda_{0,l}} v(0)\right\},$$

where $U_l(p)$ is given by (**??**). We have then two different cases depending on whether $\overrightarrow{p}$ is below or above $\widetilde{p}$.

Case 1: $\overrightarrow{p} \leq \widetilde{p}$. We can define $F_l(p)$ and $F_h(p)$ below $\overrightarrow{p}$ so that both types are indifferent for all $p \leq \overrightarrow{p}$, that is, $U_l(p) = U_l(\overrightarrow{p})$ and $U_h(p) = U_h(\overrightarrow{p})$. As a result, we end up with an equilibrium, where the low type bidding support is $[0, \overrightarrow{p}]$ and high type bidding support is $[0, p'']$.

Case 2: $\overrightarrow{p} > \widetilde{p}$. By part 1 of Lemma 14, we cannot have indifference simultaneously for both types above $\widetilde{p}$, and hence the same structure as in Case 1 is not possible. Instead, we will construct an interval $[\overleftarrow{p}, \overrightarrow{p}]$ containing $\widetilde{p}$, where only low type is active: define $F_l(p)$ within $[\overleftarrow{p}, \overrightarrow{p}]$ such that $U_l(p) = U_l(\overrightarrow{p})$ for $p \in (\overleftarrow{p}, \overrightarrow{p})$, where $U_l(p)$ is given by (**??**) with $F_h(p) = F_h(\overrightarrow{p})$. By part 2 of Lemma 14, $U_h'(p) > 0$ for $p > \widetilde{p}$ and $U_h'(p) < 0$ for $p < \widetilde{p}$. We then define $\overleftarrow{p}$ as the point where $U_h(\overleftarrow{p}) = U_h(\overrightarrow{p})$. Since $\overleftarrow{p} < \widetilde{p}$, we can define $F_l(p)$ and $F_h(p)$ below $\overleftarrow{p}$ so that both types are indifferent, that is $U_l(p) = U_l(\overleftarrow{p})$ and $U_h(p) = U_h(\overleftarrow{p})$ for all $p \leq \overleftarrow{p}$. As a result we have an equilibrium, where the low type bidding support is $[0, \overrightarrow{p}]$ and high type bidding support is disconnected and given by $[0, \overleftarrow{p}] \cup [\overrightarrow{p}, p'']$. ■

Proof of Lemma 8. If $\pi_l^*(\pi_h) = 0$, it is easy to check that (5) holds since $W(0, \pi_h)$ is concave in π_h. If $\pi_l^*(\pi_h) > 0$, the first order condition for π_l must hold:

$$\begin{aligned} 0 &= (1-\alpha) q \left(v(1) e^{-\alpha\pi_h - (1-\alpha)\pi_l^*(\pi_h)} - c\right) & (13) \\ &\quad + (1-\beta)(1-q)\left(v(0) e^{-\beta\pi_h - (1-\beta)\pi_l^*(\pi_h)} - c\right). & (14) \end{aligned}$$

If $\pi_h = \pi_h^{opt}$, then $\pi_l^*(\pi_h) = \pi_l^{opt}$ and (14) is satisfied since

$$v(1) e^{-\alpha\pi_h^{opt} - (1-\alpha)\pi_l^{opt}(\pi_h)} - c = v(0) e^{-\beta\pi_h^{opt} - (1-\beta)\pi_l^{opt}} - c = 0.$$

If $\pi_h > (<)\, \pi_h^{opt}$, we note that since $\alpha > \beta$, (14) can only hold if

$$v(1) e^{-\alpha\pi_h - (1-\alpha)\pi_l^*(\pi_h)} - c < (>)\, 0 < (>)\, v(0) e^{-\beta\pi_h - (1-\beta)\pi_l^*(\pi_h)} - c.$$

But then, since $\alpha > (1-\alpha)$ and $\beta < (1-\beta)$, we have

$$\begin{aligned}\frac{\partial W\left(\pi_l^*\left(\pi_h\right),\pi_h\right)}{\partial \pi_h} &= \alpha q\left(v\left(1\right)e^{-\alpha\pi_h-(1-\alpha)\pi_l^*(\pi_h)}-c\right) \\ &+ \beta\left(1-q\right)\left(v\left(0\right)e^{-\beta\pi_h-(1-\beta)\pi_l^*(\pi_h)}-c\right) < (>)\, 0.\end{aligned}$$

The result then follows from the envelope theorem. ■

Proof of Proposition 9. Let $c \geq \bar{c}$ and assume that entry rates are given by $\pi_h = \pi_h^{opt} > 0$ and $\pi_l = \pi_l^{opt} = 0$. It is easy to check that in all the auction forms, the payoffs to the high and low type bidders are given by $V_h^0\left(\pi_h^{opt},0\right) = c$ and $V_l^0\left(\pi_h^{opt},0\right) < c$, respectively. Hence, socially optimal entry profile is an equilibrium. It is straight-forward to show that no other equilibria exist. ■

Proof of Proposition 10. We have already shown in Proposition 4 that there cannot be atoms in the bidding distribution of an equilibrium with interim entry and hence the atomless equilibrium of Proposition 6 is the only candidate for the bidding stage. As stated in Proposition 6, zero is in the support of the low type. The upper bound b_h follows from the break-even condition of a high type bidder who by bidding the highest bid in the support wins with probability 1. For an arbitrary entry profile (π_h,π_l), denote the expected payoff of a player with signal θ in the atomless bidding equilibrium by $V_\theta^{IF}(\pi_h,\pi_l)$, $\theta \in \{h,l\}$.

Consider next the entry stage. We want to show that for each $c \in (0,\bar{c})$ there is a unique (π_h,π_l) such that $V_h^{IF}(\pi_l,\pi_h) = V_l^{IF}(\pi_l,\pi_h) = c$. To do that, we will show that the following properties hold:

P1. $V_h^{IF}(\pi_l,\pi_h)$ and $V_l^{IF}(\pi_l,\pi_h)$ are continuous in (π_l,π_h)

P2. $V_h^{IF}(\pi_l,\pi_h)$ and $V_l^{IF}(\pi_l,\pi_h)$ are strictly decreasing in π_l

P3. There are unique points $\pi_l^{h*} > \pi_l^{l*} > 0$ and $\pi_h^{l*} > \pi_h^{h*} > 0$ such that

$$\begin{aligned}V_h^{IF}\left(\pi_l^{h*},0\right) &= V_l^{IF}\left(\pi_l^{l*},0\right) = c,\\ V_h^{IF}\left(0,\pi_h^{h*}\right) &= V_l^{IF}\left(0,\pi_h^{l*}\right) = c.\end{aligned}$$

P4. For every (π_l,π_h) such that $V_h^{IF}(\pi_l,\pi_h) = V_l^{IF}(\pi_l,\pi_h) = c$, we have

$$\frac{\partial V_l^{IF}(\pi_l,\pi_h)}{\partial \pi_l}\frac{\partial V_h^{IF}(\pi_l,\pi_h)}{\partial \pi_h} > \frac{\partial V_h^{IF}(\pi_l,\pi_h)}{\partial \pi_l}\frac{\partial V_l^{IF}(\pi_l,\pi_h)}{\partial \pi_h}. \tag{15}$$

Before proving these properties, we show that they imply the existence of a unique (π_h,π_l) with $V_h^{IF}(\pi_l,\pi_h) = V_l^{IF}(\pi_l,\pi_h) = c$. Observe first that P1 - P3 imply that there are continuous and decreasing curves $\pi_l^{h*}(\pi_h)$ and $\pi_l^{l*}(\pi_h)$ such that

$$\pi_l^{h*}(0) = \pi_l^{h*},\ \pi_l^{l*}(0) = \pi_l^{l*},\ \pi_l^{h*}\left(\pi_h^{h*}\right) = 0,\ \pi_l^{l*}\left(\pi_h^{l*}\right) = 0$$

and

$$V_l^{IF}\left(\pi_l^{l*}\left(\pi_h\right),\pi_h\right) = c \text{ for all } \pi_h\in\left[0,\pi_h^{l*}\right],$$
$$V_h^{IF}\left(\pi_l^{h*}\left(\pi_h\right),\pi_h\right) = c \text{ for all } \pi_h\in\left[0,\pi_h^{h*}\right].$$

These curves must cross each other at least once at some (π_h,π_l) where $V_h^{IF}(\pi_l,\pi_h)=V_l^{IF}(\pi_l,\pi_h)=c$. We show next that they cannot cross more than once if P4 holds. Suppose that the curves cross at (π_h,π_l). Totally differentiate $V_h^{IF}(\pi_l,\pi_h)$ and consider an infinitesimal movement along $\pi_l^{h*}(\pi_h)$ in the direction where $d\pi_h>0$. Along that curve

$$\frac{\partial V_h^{IF}(\pi_l,\pi_h)}{\partial\pi_h}d\pi_h+\frac{\partial V_h^{IF}(\pi_l,\pi_h)}{\partial\pi_l}d\pi_l=0$$

so that

$$\frac{\partial V_h^{IF}(\pi_l,\pi_h)}{\partial\pi_h}/\frac{\partial V_h^{IF}(\pi_l,\pi_h)}{\partial\pi_l}=-\frac{d\pi_l}{d\pi_h}. \tag{16}$$

By Property P4, we have

$$\frac{\partial V_l^{IF}(\pi_l,\pi_h)}{\partial\pi_l}\frac{\partial V_h^{IF}(\pi_l,\pi_h)}{\partial\pi_h}>\frac{\partial V_h^{IF}(\pi_l,\pi_h)}{\partial\pi_l}\frac{\partial V_l^{IF}(\pi_l,\pi_h)}{\partial\pi_h}.$$

Since by Property P2 we have $\frac{\partial V_l^{IF}(\pi_l,\pi_h)}{\partial\pi_l}<0$ and $\frac{\partial V_h^{IF}(\pi_l,\pi_h)}{\partial\pi_l}<0$, this is equivalent to

$$\frac{\partial V_h^{IF}(\pi_l,\pi_h)}{\partial\pi_h}/\frac{\partial V_h^{IF}(\pi_l,\pi_h)}{\partial\pi_l}>\frac{\partial V_l^{IF}(\pi_l,\pi_h)}{\partial\pi_h}/\frac{\partial V_l^{IF}(\pi_l,\pi_h)}{\partial\pi_l}$$

so that combining with (16) we have

$$\frac{\partial V_l^{IF}(\pi_l,\pi_h)}{\partial\pi_h}/\frac{\partial V_l^{IF}(\pi_l,\pi_h)}{\partial\pi_l}<-\frac{d\pi_l}{d\pi_h}.$$

Since $\frac{\partial V_l^{IF}(\pi_l,\pi_h)}{\partial\pi_l}<0$ and $d\pi_h>0$, this means that

$$\frac{\partial V_l}{\partial\pi_h}d\pi_h+\frac{\partial V_l}{\partial\pi_l}d\pi_l>0,$$

and hence at any crossing point $\pi_l^{h*}(\pi_h)$ crosses $\pi_l^{l*}(\pi_h)$ from above when going in the direction of increasing π_h. Since $\pi_l^{h*}(\pi_h)$ and $\pi_l^{l*}(\pi_h)$ are continuous curves, this implies that there cannot be more than one crossing point. There is a unique (π_h,π_l) where $V_h^{IF}(\pi_l,\pi_h)=V_l^{FPA}(\pi_l,\pi_h)=c$.

The final step is to prove that properties P1 - P4 actually hold. In both of the two cases in Proposition 6, the low type payoff can be written as:

$$V_l^{IF}(\pi_l,\pi_h)=V_l^0(\pi_l,\pi_h).$$

This is clearly a continuous function and it is easy to show that

$$\frac{\partial V_l^{IF}(\pi_l, \pi_h)}{\partial \pi_l} < 0, \ \frac{\partial V_l^{IF}(\pi_l, \pi_h)}{\partial \pi_h} < 0.$$

To derive an expression for $V_h^{IF}(\pi_l, \pi_h)$, we need to consider separately the two cases of Proposition 6. Consider first the case, where 0 is in the bidding supports of both types. In that case we have

$$V_h^{IF}(\pi_l, \pi_h) = V_h^0(\pi_l, \pi_h),$$

which is a continuous function and

$$\frac{\partial V_h^{IF}(\pi_l, \pi_h)}{\partial \pi_l} < 0, \ \frac{\partial V_h^{IF}(\pi_l, \pi_h)}{\partial \pi_h} < 0.$$

It is also straightforward to show that

$$\frac{\partial V_l^0(\pi_l, \pi_h)}{\partial \pi_l} \frac{\partial V_h^0(\pi_l, \pi_h)}{\partial \pi_h} > \frac{\partial V_h^0(\pi_l, \pi_h)}{\partial \pi_l} \frac{\partial V_l^0(\pi_l, \pi_h)}{\partial \pi_h}$$

so that also property P4 holds when $V_h^{FPA}(\pi_l, \pi_h) = V_h^0(\pi_l, \pi_h) = c$ and $V_l^{FPA}(\pi_l, \pi_h) = V_l^0(\pi_l, \pi_h) = c$.

Consider next the second case, where the bidding supports of the two types do not overlap. Let $p'(\pi_h, \pi_l)$ denote the only common point in the two supports, and note that this is a function of the entry rates. Since $p'(\pi_h, \pi_l)$ is in the support of both type of players' bidding strategy, we can write the payoffs of the two types as

$$V_\theta^{IF}(\pi_h, \pi_l) = q_\theta w_1(\pi_h, \pi_l) + (1 - q_\theta) w_0(\pi_h, \pi_l), \ \theta = h, l,$$

where

$$q_h = \frac{\alpha q}{\alpha q + \beta(1-q)} \text{ and } q_l = \frac{\beta(1-q)}{\alpha q + \beta(1-q)}$$

are the posteriors of the two types, and $w_\omega(\pi_h, \pi_l)$, $\omega = 0, 1$, is the payoff from bidding $p'(\pi_h, \pi_l)$ conditional on state. By bidding $p'(\pi_h, \pi_l)$, a player wins if and only if there are no high type entrants, and therefore we may write the state conditional payoffs from this strategy as:

$$\begin{aligned} w_1(\pi_l, \pi_h) &= e^{-\alpha\pi_h}(v(1) - p'(\pi_l, \pi_h)), \\ w_0(\pi_l, \pi_h) &= e^{-\beta\pi_h}(v(0) - p'(\pi_l, \pi_h)), \end{aligned} \tag{17}$$

where $p'(\pi_l, \pi_h)$ can be solved from the indifference condition for a low type bidder between bidding 0 and bidding $p'(\pi_l, \pi_h)$:

$$\begin{aligned} & q_l e^{-\alpha\pi_h - (1-\alpha)\pi_l} v(1) + (1 - q_l) e^{-\beta\pi_h - (1-\beta)\pi_l} v(0) \\ = \ & q_l e^{-\alpha\pi_h}(v(1) - p'(\pi_l, \pi_h)) + (1 - q_l) e^{-\beta\pi_h}(v(0) - p'(\pi_l, \pi_h)), \end{aligned}$$

which gives

$$p'(\pi_l,\pi_h)=\frac{q_l e^{-\alpha\pi_h}\left(1-e^{-(1-\alpha)\pi_l}\right)v(1)+(1-q_l)\,e^{-\beta\pi_h}\left(1-e^{-(1-\beta)\pi_l}\right)v(0)}{q_l e^{-\alpha\pi_h}v(1)+(1-q_l)\,e^{-\beta\pi_h}v(0)}.$$

This is continuous in (π_h,π_l), and so are $V_\theta^{IF}(\pi_h,\pi_l)$, $\theta=h,l$. From this we can also show that

$$\begin{aligned}\frac{\partial p'(\pi_l,\pi_h)}{\partial\pi_l} &> 0,\\ \frac{\partial p'(\pi_l,\pi_h)}{\partial\pi_h} &< 0.\end{aligned}$$

Differentiating $w_\omega(\pi_l,\pi_h)$, we have

$$\begin{aligned}\frac{\partial w_1(\pi_l,\pi_h)}{\partial\pi_h} &= -e^{-\alpha\pi_h}\left(\alpha\left(v(1)-p'(\pi_l,\pi_h)\right)+\frac{\partial p'(\pi_l,\pi_h)}{\partial\pi_h}\right),\\ \frac{\partial w_0(\pi_l,\pi_h)}{\partial\pi_h} &= -e^{-\beta\pi_h}\left(\beta\left(v(0)-p'(\pi_l,\pi_h)\right)+\frac{\partial p'(\pi_l,\pi_h)}{\partial\pi_h}\right),\\ \frac{\partial w_1(\pi_l,\pi_h)}{\partial\pi_l} &= -e^{-\alpha\pi_h}\frac{\partial p'(\pi_l,\pi_h)}{\partial\pi_l},\\ \frac{\partial w_0(\pi_l,\pi_h)}{\partial\pi_l} &= -e^{-\beta\pi_h}\frac{\partial p'(\pi_l,\pi_h)}{\partial\pi_l},\end{aligned}$$

and we see immediately that

$$\frac{\partial w_1(\pi_l,\pi_h)}{\partial\pi_l}<0,\ \frac{\partial w_0(\pi_l,\pi_h)}{\partial\pi_l}<0, \tag{18}$$

which implies that also in this case we have

$$\frac{\partial V_h^{IF}(\pi_l,\pi_h)}{\partial\pi_l}<0.$$

To check property P3, note first that since $V_l^{IF}(\pi_h,\pi_l)=V_l^0(\pi_h,\pi_l)$ for all (π_h,π_l), we have

$$\begin{aligned}V_l^{IF}(\pi_h,0) &= V_l^0(\pi_h,0)\ \text{and}\\ V_l^{IF}(0,\pi_l) &= V_l^0(0,\pi_l).\end{aligned}$$

For the high type, it is easy to show that if there are no low types, the payoff of high type is equal to the social contribution and

$$V_h^{IF}(\pi_h,0)=V_h^0(\pi_h,0).$$

Moreover, a high type always gets at least the social contribution so that

$$V_h^{IF}(0,\pi_l)\ge V_h^0(0,\pi_l).$$

It is easy to check using these that P3 holds whenever $c < \bar{c}$.

Finally, we consider property P4 in the case where bidding supports do not overlap. First, we can write (15) as:

$$\begin{aligned} & \left(q_l \frac{\partial w_1(\pi_l,\pi_h)}{\partial \pi_l} + (1-q_l)\frac{\partial w_0(\pi_l,\pi_h)}{\partial \pi_l}\right)\left(q_h \frac{\partial w_1(\pi_l,\pi_h)}{\partial \pi_h} + (1-q_h)\frac{\partial w_0(\pi_l,\pi_h)}{\partial \pi_h}\right) \\ > & \left(q_h \frac{\partial w_1(\pi_l,\pi_h)}{\partial \pi_l} + (1-q_h)\frac{\partial w_0(\pi_l,\pi_h)}{\partial \pi_l}\right)\left(q_l \frac{\partial w_1(\pi_l,\pi_h)}{\partial \pi_h} + (1-q_l)\frac{\partial w_0(\pi_l,\pi_h)}{\partial \pi_h}\right). \end{aligned}$$

By straightforward algebra, this can be written as

$$\begin{aligned} & q_l\left(\frac{\partial w_1(\pi_l,\pi_h)}{\partial \pi_l}\right)\left(\frac{\partial w_0(\pi_l,\pi_h)}{\partial \pi_h}\right) + q_h\left(\frac{\partial w_0(\pi_l,\pi_h)}{\partial \pi_l}\right)\left(\frac{\partial w_1(\pi_l,\pi_h)}{\partial \pi_h}\right) \\ > \; & q_h\left(\frac{\partial w_1(\pi_l,\pi_h)}{\partial \pi_l}\right)\left(\frac{\partial w_0(\pi_l,\pi_h)}{\partial \pi_h}\right) + q_l\left(\frac{\partial w_0(\pi_l,\pi_h)}{\partial \pi_l}\right)\left(\frac{\partial w_1(\pi_l,\pi_h)}{\partial \pi_h}\right). \end{aligned}$$

Since $q_h > q_l$, we see that (15) is equivalent to

$$\frac{\partial w_1(\pi_l,\pi_h)}{\partial \pi_h}\frac{\partial w_0(\pi_l,\pi_h)}{\partial \pi_l} > \frac{\partial w_1(\pi_l,\pi_h)}{\partial \pi_l}\frac{\partial w_0(\pi_l,\pi_h)}{\partial \pi_h}. \tag{19}$$

Differentiating (17), we have:

$$\begin{aligned} & \frac{\partial w_1(\pi_l,\pi_h)}{\partial \pi_h}\frac{\partial w_0(\pi_l,\pi_h)}{\partial \pi_l} \\ = \; & e^{-(\alpha-\beta)\pi_h}\left(\alpha\left(v(1) - p'(\pi_l,\pi_h)\right) + \frac{\partial p'(\pi_l,\pi_h)}{\partial \pi_h}\right)\frac{\partial p'(\pi_l,\pi_h)}{\partial \pi_l}, \\ & \frac{\partial w_0(\pi_l,\pi_h)}{\partial \pi_h}\frac{\partial w_1(\pi_l,\pi_h)}{\partial \pi_l} \\ = \; & e^{-(\alpha-\beta)\pi_h}\left(\beta\left(v(0) - p'(\pi_l,\pi_h)\right) + \frac{\partial p'(\pi_l,\pi_h)}{\partial \pi_h}\right)\frac{\partial p'(\pi_l,\pi_h)}{\partial \pi_l}, \end{aligned}$$

from which we can check that (19) holds, which is equivalent to (15). ■

Proof of Proposition 11. Suppose that there is an equilibrium with $\pi_m > 0$ for some $m \in \{2, ..., M-1\}$. Let V_ω, $\omega = 0, 1$, denote the ex-ante equilibrium payoff conditional on state for bidding according to type m strategy. Since m breaks just even in equilibrium, $q_m V_1 + (1-q_m) V_0 = c$. We have $q_1 < q_m < q_M$. Hence, if $V_1 > V_0$, then type M can get strictly more than c by mimicking the bidding strategy of m, and if $V_0 > V_1$, then type 1 can get strictly more than c by mimicking the bidding strategy of m. Therefore, it must be that $V_0 = V_1$. This of course means that any type can get c by mimicking type m bidding strategy. It remains to show that some type gets strictly more by deviating from m's bidding strategy.

Note that since we consider formal auctions, the bidding strategy is conditional on n, the realized number of entrants. Let $R_{\omega}^{n}(p)$ denote the expected payoff of bidding p, conditional on n and conditional on ω. Suppose that there is some n and either some interval (p', p'') or an atom p''' in the support of m's bidding strategy such that $R_1^n(p) \neq R_0^n(p)$ within (p', p'') or at p'''. Then, either type 1 or type M gets more in expectation more than m by deviating to (p', p'') or p''' for this n. By using this deviation for n, and by mimicking m for all other n, this type (either 1 or M) gets a strictly higher ex-ante payoff than m, contradicting the assumption that this strategy is an equilibrium in the overall game with interim entry. It follows that unless $R_1^n(p) = R_0^n(p)$ within the whole support of type m for all n, there cannot exist an equilibrium with $\pi_m > 0$, $m \in \{2, ..., M-1\}$.

It remains to show that we cannot have $R_1^n(p) = R_0^n(p)$ within the whole support of type m for all n. First, note that for any $n \geq 2$, $\underline{p}_n := \min \bigcup_{i=1}^{M} \text{supp} F_i^n > v(0)$, where F_i^n is the bidding distribution of type i for realized number of entrants n. If this were not the case, then anyone bidding $\underline{p}_n$ would have a strictly profitable deviation up (recall that $n \geq 2$, so if there is no atom at $\underline{p}_n$, then one can never win by bidding there, and if there is an atom, a small deviation increases probability of winning discretely). But for all $p > v(0)$, we always have $R_0^n(p) < 0$, and hence $R_1^n(p) \neq R_0^n(p)$. ■

Proof of Proposition 12. The entry rates of the low type bidders conditional on state are $\lambda_{0,l} := (1-\beta)\pi_l$ and $\lambda_{1,l} := (1-\alpha)\pi_l$. Note that

$$\frac{1 - e^{-(1-\beta)\pi_l}}{1 - e^{-(1-\alpha)\pi_l}}$$

is decreasing in π_l with $\lim_{\pi_l \to \infty} \frac{1-e^{-(1-\beta)\pi_l}}{1-e^{-(1-\alpha)\pi_l}} = 1$ and $\lim_{\pi_l \to 0} \frac{1-e^{-(1-\beta)\pi_l}}{1-e^{-(1-\alpha)\pi_l}} = \frac{1-\beta}{1-\alpha}$. Hence, if

$$\frac{1-\beta}{1-\alpha} > \frac{v(1)}{v(0)}$$

and π_l is small enough, we have

$$\frac{1 - e^{-\lambda_{0,l}}}{1 - e^{-\lambda_{1,l}}} > \frac{v(1)}{v(0)},$$

so by Proposition 6, zero is in the support of both types of bidders. In this case, the payoff in the auction is $V_{\theta}^{0}(\pi_h, \pi_l)$ and equilibrium must be socially optimal:

$$\begin{aligned} \pi_h &= \pi_h^{opt} = \frac{1}{\alpha - \beta}\left((1-\beta)\log\left(\frac{v(1)}{c}\right) - (1-\alpha)\log\left(\frac{v(0)}{c}\right)\right), \\ \pi_l &= \pi_l^{opt} = \frac{1}{\alpha - \beta}\left(-\beta\log\left(\frac{v(1)}{c}\right) + \alpha\log\left(\frac{v(0)}{c}\right)\right). \end{aligned}$$

Clearly this must be the case when c is sufficiently close to $\bar{c}$ (when $c = \bar{c}$, we have $\pi_l^{opt} = 0$). In all the other auction formats, at least the high type bidder gets strictly more than $V_h^0(\pi_h, \pi_l)$, and hence $\pi_h > \pi_h^*(\pi_l)$, so that equilibrium is inefficient and generates a strictly lower total surplus than informal FPA. ■

Proof of Proposition 13. As a first step, we show that with an exogenous entry profile (π_h, π_l) a high type gets a higher payoff in formal auctions than in informal first-price auction if π_l is small enough.

If zero is in the bidding support of both types for informal first-price auction, the result is immediate. Therefore, assume there is no overlap in the bidding supports in informal first-price auction and denote by p' the common point in the two supports.

Let us contrast informal first-price auction to formal second-price auction, and consider the following bidding strategy. In informal first-price auction, let both types bid p'. Since this is in the support of both types, it generates the equilibrium payoff to both types. In formal second-price auction, let both types bid $\underline{p}(n) + \varepsilon$, where

$$\underline{p}(n) = E\left[v \,\middle|\, \theta = l,\, N^l = n-1, N^h = 0\right]$$

is the equilibrium pooling bid for the low type and $\varepsilon > 0$ is some number small enough so that $\underline{p}(n) + \varepsilon$ is strictly below the lowest point in the high type bidding support. With this strategy, a bidder wins if and only if there are no (other) high type bidders, and price in such a case is either zero or $\underline{p}(n)$. Clearly, this strategy is a weakly best-response for both types. Since using these strategies, a player gets the same allocation in both auction formats (get the object if and only if there are no high types present), we may compare the players' preference over the auction formats by contrasting their expected payment conditional on winning across the auction formats.

Start with the low type. Since the low type gets expected payoff zero in both auction formats, she is indifferent between bidding p' in informal first-price auction and bidding $\underline{p}(n) + \varepsilon$ in formal second-price auction. Therefore, the expected payment conditional on winning must be the same in the two cases, leading to:

$$\begin{aligned} p' &= E\left(p \,\middle|\, \theta = l,\ \text{"win by bidding } \underline{p}(n) + \varepsilon\text{"}\right) \\ &= \Pr\left(N^l = 0 \,\middle|\, \theta = l, N^h = 0\right) \cdot 0 \\ &\quad + \sum_{k=1}^{\infty} \Pr\left(N^l = k \,\middle|\, \theta = l, N^h = 0\right) \underline{p}(k+1). \end{aligned}$$

Consider next the high type. Her expected payment when bidding $\underline{p}(n) + \varepsilon$ in formal second-price auction is

$$
\begin{aligned}
& E\left(p \,\middle|\, \theta = h,\ \text{"win by bidding } \underline{p}(n)+\varepsilon\text{"}\right) \\
= \ & \sum_{k=1}^{\infty} \Pr\left(N^l = k \,\middle|\, \theta = h, N^h = 0\right) \underline{p}(k+1) \\
= \ & \sum_{k=1}^{\infty} \Pr\left(N^l = k \,\middle|\, \theta = h, N^h = 0\right) E\left(v \,\middle|\, \theta = l, N^h = 0, N^l = k\right)
\end{aligned}
$$

and hence she prefers the formal second-price auction if

$$
\begin{aligned}
& \sum_{k=1}^{\infty} \Pr\left(N^l = k \,\middle|\, \theta = h, N^h = 0\right) \underline{p}(k+1) \\
< \ & p' = \sum_{k=1}^{\infty} \Pr\left(N^l = k \,\middle|\, \theta = l, N^h = 0\right) \underline{p}(k+1), \qquad (20)
\end{aligned}
$$

where

$$\underline{p}(k+1) := E\left(v \,\middle|\, \theta = l, N^h = 0, N^l = k\right)$$

is a decreasing function of k (for $k \geq 1$). Note that the only difference in the two formulas in (20) is that the probability

$$\Pr\left(N^l = k \,\middle|\, \theta, N^h = 0\right)$$

is conditioned on $\theta = h$ in the first line and $\theta = l$ in the second line. Since a high signal makes state $\omega = 1$ more likely, a simple sufficient condition for (20) to hold is that

$$\Pr\left(N^l = k \,\middle|\, \omega = 1, N^h = 0\right) < \Pr\left(N^l = k \,\middle|\, \omega = 0, N^h = 0\right)$$

for all $k \geq 1$. Since

$$N^l \,\middle|\, \left(\omega = 1, N^h = 0\right) \sim \mathit{Poisson}\left(\lambda_{\omega,l}\right),$$

we know that (20) holds if

$$\lambda_{0,l} e^{-\lambda_{0,l}} > \lambda_{1,l} e^{-\lambda_{1,l}},$$

that is

$$(1-\beta)\,\pi_l\, e^{-(1-\beta)\pi_l} > (1-\alpha)\,\pi_l\, e^{-(1-\alpha)\pi_l}$$

or

$$\pi_l < \frac{\log(1-\beta) - \log(1-\alpha)}{\alpha - \beta}.$$

Therefore, a high-type bidder has a lower expected payment in the formal second-price auction for π_l small enough, which means that her expected payoff is higher in that auction formal. Since low type is always indifferent, this means that that expected revenue for the seller is higher in the informal first-price auction

The second step is just to conclude that a change of auction format from informal first-price auction to a formal auction increases π_h, which is already too high in the informal first-price auction. Since in both auction formats, $\pi_l = \pi_l^* (\pi_h)$, this further distortion will move the equilibrium entry point along $\pi_l^* (\pi_h)$ further away from social optimum, which by Lemma 8 decreases social surplus and hence revenue. ■

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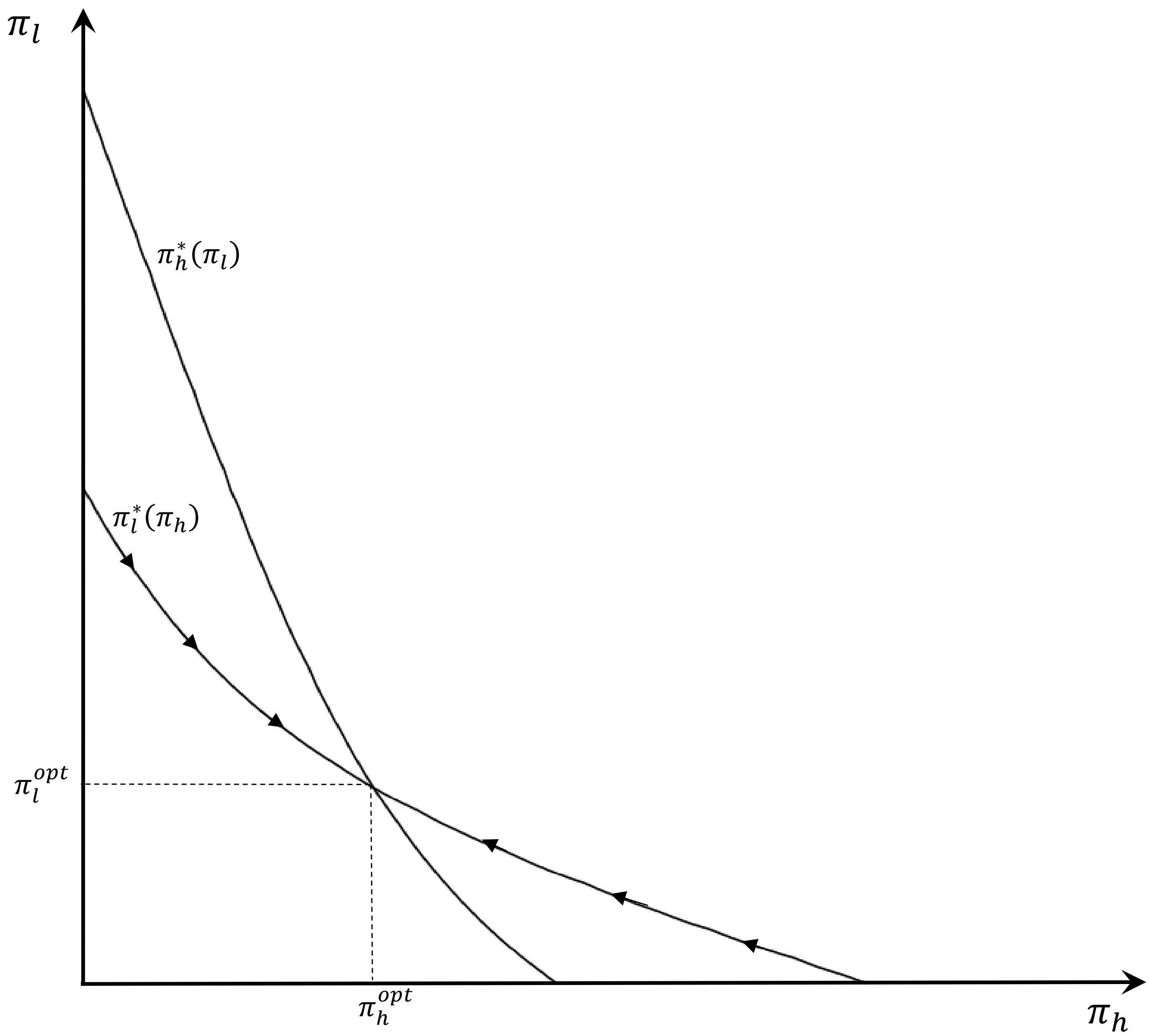

Figure 1: Social planner’s reaction curves.